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Fermion texture and sterile neutrinos

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Abstract

An explicit form of charged–lepton mass matrix, predicting $m_\tau = 1776.80$ MeV from the experimental values of m_e and m_μ (in good agreement with the experimental figure $m_\tau = 1777.05^{+0.29}_{-0.26}$ MeV), is applied to three neutrinos ν_e, ν_μ, ν_τ in order to correlate tentatively their masses and mixing parameters. It is suggested that for neutrinos the diagonal elements of the mass matrix are small *versus* its off-diagonal elements. Under such a conjecture, the neutrino masses, lepton Cabibbo—Kobayashi—Maskawa matrix and neutrino oscillation probabilities are calculated in the corresponding lowest perturbative order. Then, the nearly maximal mixing of ν_μ and ν_τ is predicted in consistency with the observed deficit of atmospheric ν_μ's. However, the predicted deficit of solar ν_e's is much too small to explain the observed effect, what suggests the existence of (at least) one sort, $\nu_s^{(e)}$, of sterile neutrinos, whose mixing with ν_e would be responsible for the observed deficit. Perspectives for applying the same form of mass matrix to quarks are also outlined. Two independent predictions of $|V_{ub}|/|V_{cb}| = 0.0753 \pm 0.0032$ and unitary angle $\gamma \simeq 70^\circ$ are deduced from the experimental values of $|V_{us}|$ and $|V_{cb}|$ (with the use of quark masses m_s, m_c and m_b). In the last three Sections, the option of two sorts, $\nu_s^{(e)}$ and $\nu_s^{(\mu)}$, of sterile neutrinos is considered. They may dominate neutrino mixing, and even cause that two extra neutrino mass states (arising then) are agents of some tiny neutrino instability and related damping of ν_e and ν_μ oscillations. In Appendix, three conventional Majorana sterile neutrinos are discussed.

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1. Introduction

In this paper, the explicit form of mass matrix invented for three generations of charged leptons e^-, μ^-, τ^-, and being surprisingly good for their masses [1], is applied to three generations of neutrinos ν_e, ν_μ, ν_τ, in order to correlate tentatively their masses and mixing parameters. This form reads

$$\left(M^{(f)}_{\alpha\beta}\right) = \frac{1}{29}\left(\begin{array}{ccc} \mu^{(f)}\varepsilon^{(f)} & 2\alpha^{(f)}e^{i\varphi^{(f)}} & 0 \\ 2\alpha^{(f)}e^{-i\varphi^{(f)}} & 4\mu^{(f)}(80+\varepsilon^{(f)})/9 & 8\sqrt{3}\,\alpha^{(f)}e^{i\varphi^{(f)}} \\ 0 & 8\sqrt{3}\,\alpha^{(f)}e^{-i\varphi^{(f)}} & 24\mu^{(f)}(624+\varepsilon^{(f)})/25 \end{array}\right)\,, \qquad (1)$$

where the label $f = \nu,\, e$ denotes neutrinos and charged leptons, respectively, while $\mu^{(f)}$, $\varepsilon^{(f)}$, $\alpha^{(f)}$ and $\varphi^{(f)}$ are real constants to be determined from the present and future experimental data for lepton masses and mixing parameters ($\mu^{(f)}$ and $\alpha^{(f)}$ are mass–dimensional). In our approach, neutrinos are assumed to carry pure Dirac masses.

Here, the form (1) of mass matrices $\left(M^{(\nu)}_{\alpha\beta}\right)$ and $\left(M^{(e)}_{\alpha\beta}\right)$ may be considered as a detailed ansatz to be compared with the lepton data. However, in the past, we have presented an argument [2,1] in favour of the form (1), based on: (*i*) Kähler–like generalized Dirac equations (interacting with the Standard Model gauge bosons) whose *a priori* infinite series is necessarily reduced (in the case of fermions) to three Dirac equations, due to an intrinsic Pauli principle, and (*ii*) an ansatz for the fermion mass matrix, suggested by the above three–generation characteristics (*i*).

In the case of charged leptons, assuming that the off–diagonal elements of the mass matrix $\left(M^{(e)}_{\alpha\beta}\right)$ can be treated as a small perturbation of its diagonal terms (*i.e.*, that $\alpha^{(e)}/\mu^{(e)}$ is small enough), we calculate in the lowest perturbative order [1]

$$\begin{aligned} m_\tau &= \left[1776.80 + 10.2112\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\right]\ \mathrm{MeV}\,, \\ \mu^{(e)} &= 85.9924\ \mathrm{MeV} + O\left[\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\right]\mu^{(e)}\,, \\ \varepsilon^{(e)} &= 0.172329 + O\left[\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\right]\,, \qquad (2) \end{aligned}$$

when the experimental values of m_e and m_μ [3] are used as inputs. In Eqs. (2), the first terms are given as $\overset{\circ}{m}_\tau = 6(351 m_\mu - 136 m_e)/125$, $\overset{\circ}{\mu}{}^{(e)} = 29(9m_\mu - 4m_e)/320$ and $\overset{\circ}{\varepsilon}{}^{(e)} = 320 m_e/(9m_\mu - 4m_e)$, respectively. We can see that the predicted value of m_τ agrees very well with its experimental figure $m_\tau^{\rm exp} = 1777.05^{+0.29}_{-0.26}$ MeV [3], even in the zero perturbative order. To estimate $\left(\alpha^{(e)}/\mu^{(e)}\right)^2$, we can take this experimental figure as another input, obtaining

$$\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2 = 0.024^{+0.028}_{-0.025} \,, \tag{3}$$

which value is not inconsistent with zero. Hence, $\alpha^{(e)\,2} = 180^{+210}_{-190}\,{\rm MeV}^2$ due to Eq. (2).

For the unitary matrix $\left(U^{(e)}_{\alpha\beta}\right)$, diagonalizing the charged–lepton mass matrix $\left(M^{(e)}_{\alpha\beta}\right)$ according to the relation $U^{(e)\,\dagger}\, M^{(e)}\, U^{(e)} = {\rm diag}(m_e\,,\, m_\mu\,,\, m_\tau)$, we get in the lowest perturbative order

$$\left(U^{(e)}_{\alpha\beta}\right) = \left(\begin{array}{ccc} 1 - \frac{2}{29^2}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2 & \frac{2}{29}\frac{\alpha^{(e)}}{m_\mu} e^{i\varphi^{(e)}} & \frac{16\sqrt3}{29^2}\left(\frac{\alpha^{(e)}}{m_\tau}\right)^2 e^{2i\varphi^{(e)}} \\ -\frac{2}{29}\frac{\alpha^{(e)}}{m_\mu} e^{-i\varphi^{(e)}} & 1 - \frac{2}{29^2}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2 - \frac{96}{29^2}\left(\frac{\alpha^{(e)}}{m_\tau}\right)^2 & \frac{8\sqrt3}{29}\frac{\alpha^{(e)}}{m_\tau} e^{i\varphi^{(e)}} \\ \frac{16\sqrt3}{29^2}\frac{\alpha^{(e)\,2}}{m_\mu\, m_\tau}\, e^{-2i\varphi^{(e)}} & -\frac{8\sqrt3}{29}\frac{\alpha^{(e)}}{m_\tau} e^{-i\varphi^{(e)}} & 1 - \frac{96}{29^2}\left(\frac{\alpha^{(e)}}{m_\tau}\right)^2 \end{array}\right) . \tag{4}$$

2. Neutrino masses and mixing parameters

In the case of neutrinos, because of their expected tiny mass scale $\mu^{(\nu)}$, we will tentatively conjecture that the diagonal elements of the mass matrix $\left(M^{(\nu)}_{\alpha\beta}\right)$ can be treated as a small perturbation of its off–diagonal terms (*i.e.*, that $\mu^{(\nu)}/\alpha^{(\nu)}$ is small enough). In addition, we put $\varepsilon^{(\nu)} = 0$ *i.e.*, $M^{(\nu)}_{11} = 0$. Then, we calculate in the lowest perturbative order the following neutrino masses:

$$\begin{aligned} m_{\nu_1} &= \frac{|M^{(\nu)}_{12}|^2\, M^{(\nu)}_{33}}{|M^{(\nu)}_{12}|^2 + |M^{(\nu)}_{23}|^2} = \frac{1}{49} M^{(\nu)}_{33} = \frac{1}{49}\xi |M^{(\nu)}_{12}| \,, \\ m_{\nu_2,\,\nu_3} &= \mp\sqrt{|M^{(\nu)}_{12}|^2 + |M^{(\nu)}_{23}|^2} + \frac{1}{2}\left(\frac{48}{49} M^{(\nu)}_{33} + M^{(\nu)}_{22}\right) \\ &= \left[\mp 7 + \frac{1}{2}\left(\frac{48}{49}\xi + \chi\right)\right] |M^{(\nu)}_{12}| \,, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \xi &\equiv \frac{M_{33}^{(\nu)}}{|M_{12}^{(\nu)}|} = \frac{7488}{25}\frac{\mu^{(\nu)}}{\alpha^{(\nu)}} = 299.52\frac{\mu^{(\nu)}}{\alpha^{(\nu)}} \;, \\ \chi &\equiv \frac{M_{22}^{(\nu)}}{|M_{12}^{(\nu)}|} = \frac{160}{9}\frac{\mu^{(\nu)}}{\alpha^{(\nu)}} = \frac{125}{2106}\xi = \frac{1}{16.848}\xi \;, \end{aligned} \tag{6}$$

are relatively small by our perturbative conjecture, while

$$|M_{12}^{(\nu)}| = \frac{2}{29}\alpha^{(\nu)} \;,\; |M_{23}^{(\nu)}| = \frac{8\sqrt{3}}{29}\alpha^{(\nu)} = \sqrt{48}|M_{12}^{(\nu)}| \;. \tag{7}$$

As seen from Eqs. (5), the actual perturbative parameters are not ξ and χ, but rather $\xi/7$ and $\chi/7$, what is confirmed later in Eqs. (9). Note that $m_{\nu_2} < 0$, the minus sign being irrelevant in the relativistic case, where only $m^2_{\nu_2}$ is measured (*cf.* Dirac equation): $|m_{\nu_2}|$ may be considered as a phenomenological mass of ν_2.

Using Eqs. (5), we can write the formula

$$m^2_{\nu_3} - m^2_{\nu_2} = 14\left(\frac{48}{49}\xi + \chi\right)|M_{12}^{(\nu)}|^2 = 20.721\,\alpha^{(\nu)}\mu^{(\nu)} \;, \tag{8}$$

which will enable us to determine the product $\alpha^{(\nu)}\mu^{(\nu)}$ from the observed deficit of atmospheric neutrinos ν_μ, if $\nu_\mu \to \nu_\tau$ oscillations are really responsible for this effect.

We calculate also the unitary matrix $\left(U_{\alpha i}^{(\nu)}\right)$ diagonalizing the neutrino mass matrix $\left(M_{\alpha\beta}^{(\nu)}\right)$ according to the relation $U^{(\nu)\dagger}M^{(\nu)}U^{(\nu)} = \mathrm{diag}(m_{\nu_1}\,,\, m_{\nu_2}\,,\, m_{\nu_3})$. In the lowest perturbative order we obtain

$$\begin{aligned} U_{11}^{(\nu)} &= \sqrt{\frac{48}{49}}\left[1 - \left(\frac{24}{49^3} - \frac{1}{49^4}\right)\xi^2\right] \;, \\ U_{21}^{(\nu)} &= \frac{1}{49}\sqrt{\frac{48}{49}}\xi e^{-i\varphi^{(\nu)}} \;, \\ U_{31}^{(\nu)} &= -\frac{1}{7}\left[1 - \left(\frac{73}{49^3} - \frac{1}{49^4}\right)\xi^2 + \frac{1}{49}\xi\chi\right] e^{-2i\varphi^{(\nu)}} \;, \\ U_{12}^{(\nu)} &= -\frac{1}{\sqrt{2}}\frac{1}{7}\left(1 + \frac{36}{7\cdot 49}\xi + \frac{1}{28}\chi\right) e^{i\varphi^{(\nu)}} \;, \\ U_{22}^{(\nu)} &= \frac{1}{\sqrt{2}}\left(1 + \frac{12}{7\cdot 49}\xi - \frac{1}{28}\chi\right) \;, \\ U_{32}^{(\nu)} &= -\frac{1}{\sqrt{2}}\sqrt{\frac{48}{49}}\left(1 - \frac{13}{7\cdot 49}\xi + \frac{1}{28}\chi\right) e^{-i\varphi^{(\nu)}} \;, \end{aligned}$$

$$
\begin{aligned}
U_{13}^{(\nu)} &= \frac{1}{\sqrt{2}}\frac{1}{7}\left(1-\frac{36}{7\cdot 49}\xi-\frac{1}{28}\chi\right) e^{2i\varphi^{(\nu)}} \;, \\
U_{23}^{(\nu)} &= \frac{1}{\sqrt{2}}\left(1-\frac{12}{7\cdot 49}\xi+\frac{1}{28}\chi\right) e^{i\varphi^{(\nu)}} \;, \\
U_{33}^{(\nu)} &= \frac{1}{\sqrt{2}}\sqrt{\frac{48}{49}}\left(1+\frac{13}{7\cdot 49}\xi-\frac{1}{28}\chi\right)
\end{aligned}
\tag{9}
$$

with $\chi = (125/2106)\xi = \xi/16.848$.

Denoting by $\nu_\alpha = \nu_e\,,\,\nu_\mu\,,\,\nu_\tau$ and $\nu_i = \nu_1\,,\,\nu_2\,,\,\nu_3$ the flavor and mass neutrino fields, respectively, we have the unitary transformation

$$
\nu_\alpha = \sum_i \left(V^\dagger\right)_{\alpha\, i} \nu_i = \sum_i V^*_{i\,\alpha}\nu_i \;, \tag{10}
$$

where the lepton counterpart $(V_{i\,\alpha})$ of the Cabibbo—Kobayashi—Maskawa matrix is given as $V = U^{(\nu)\,\dagger}U^{(e)} \simeq U^{(\nu)\,\dagger}$ or

$$
V_{i\,\alpha} = \sum_\beta \left(U^{(\nu)\,\dagger}\right)_{i\,\beta} U^{(e)}_{\beta\alpha} \simeq U^{(\nu)\,*}_{\alpha\, i} \;, \tag{11}
$$

the approximate equality being valid for negligible $\alpha^{(e)}/\mu^{(e)}$ when $U^{(e)}_{\beta\alpha} \simeq \delta_{\beta\alpha}$ due to Eq. (4). Of course, in Eqs. (9) we wrote $\alpha = 1, 2, 3$ for simplicity. From Eq. (10), we get the unitary transformation $|\nu_\alpha\rangle = \sum_i |\nu_i\rangle V_{i\,\alpha}$, where $|\nu_\alpha\rangle = \nu^\dagger_\alpha|0\rangle$ and $|\nu_i\rangle = \nu^\dagger_i|0\rangle$ are flavor and mass neutrino states*.

In the limit of $\mu^{(\nu)} \to 0$ (implying $\xi \to 0$ and $\chi \to 0$), we obtain from Eqs. (10), (11) and (9) the following unperturbed mixing formulae for $\nu_1\,,\,\nu_2\,,\,\nu_3$:

$$
\begin{aligned}
\nu_e &\to \frac{1}{7}\left[\sqrt{48}\nu_1 e^{-i\varphi^{(\nu)}} - \frac{1}{\sqrt{2}}\left(\nu_2 - \nu_3 e^{i\varphi^{(\nu)}}\right)\right] e^{i\varphi^{(\nu)}} \;, \\
\nu_\mu &\to \frac{1}{\sqrt{2}}\left(\nu_2 + \nu_3 e^{i\varphi^{(\nu)}}\right) \;, \\
\nu_\tau &\to -\frac{1}{7}\left[\nu_1 e^{-i\varphi^{(\nu)}} + \sqrt{\frac{48}{2}}\left(\nu_2 - \nu_3 e^{i\varphi^{(\nu)}}\right)\right] e^{-i\varphi^{(\nu)}} \;.
\end{aligned}
\tag{12}
$$

These display the maximal mixing between ν_2 and ν_3 in all three cases and a smaller mixing of $\left[\nu_2 - \nu_3 \exp\left(i\varphi^{(\nu)}\right)\right]/\sqrt{2}$ with ν_1 in the cases of ν_e and ν_τ, giving a minor admixture to ν_e and a dominating admixture to ν_τ (in ν_μ there is no admixture of ν_1).

*In place of $\nu_i = \sum_\alpha V_{i\,\alpha}\nu_\alpha$ one might use the notation $\nu'_\alpha = \sum_\beta V_{\alpha\,\beta}\nu_\beta$, analoguous to $d'_\alpha = \sum_\beta V_{\alpha\beta} d_\beta$ customary in the case of quarks where $V_{\alpha\beta} = \sum_\gamma \left(U^{(u)\,\dagger}\right)_{\alpha\,\gamma} U^{(d)}_{\gamma\,\beta}$.

3. Neutrino oscillations

Once knowing the elements $V_{i\,\alpha}$ of the lepton Cabibbo—Kobayashi—Maskawa matrix, we can calculate the probabilities of neutrino oscillations $\nu_\alpha \rightarrow \nu_\beta$ (in the vacuum) making use of the general formula

$$P(\nu_\alpha \rightarrow \nu_\beta) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = \sum_{i\,j} V_{j\,\beta} V^*_{j\,\alpha} V^*_{i\,\beta} V_{i\,\alpha} e^{2ix_{j\,i}} \;, \tag{13}$$

where $|\nu_\alpha(t)\rangle = \exp(-iHt)|\nu_\alpha\rangle$ and

$$x_{j\,i} = 1.26693\,\Delta m^2_{j\,i}\,L/E \;\;,\;\; \Delta m^2_{j\,i} = m^2_{\nu_j} - m^2_{\nu_i} \;\;, \tag{14}$$

if $\Delta m^2_{j\,i}$, L and E are measured in eV^2, km and GeV, respectively, with $L = t$ and $E = |\vec{p}|$ $(c = 1 = \hbar)$ denoting the experimental baseline and neutrino energy.

It is not difficult to show that for the mass matrix $\left(M^{(\nu)}_{\alpha\beta}\right)$, as it is given in Eq. (1), the quartic products of $V_{i\,\alpha}$'s in Eq. (13) are always real (for any phase $\varphi^{(\nu)}$), if only $V_{i\,\alpha} = U^{(\nu)\,*}_{\alpha\,i}$ (*i.e.*, $U^{(e)}_{\beta\alpha} = \delta_{\beta\alpha}$). This implies that $P(\nu_\alpha \rightarrow \nu_\beta) = P(\nu_\beta \rightarrow \nu_\alpha)$. In general, the last relation is valid in the case of CP invariance which, under the CPT theorem, provides the time–reversal invariance. Because of the real values of quartic products of $V_{i\,\alpha}$'s, the formula (13) can be rewritten as

$$P(\nu_\alpha \rightarrow \nu_\beta) = \delta_{\beta\alpha} - 4 \sum_{i<j} V_{j\,\beta} V^*_{j\,\alpha} V^*_{i\,\beta} V_{i\,\alpha} \sin^2 x_{j\,i} \tag{15}$$

without the necessity of introducing phases of these products.

With the lowest–order perturbative expressions (9) for $V_{i\,\alpha} = U^{(\nu)\,*}_{\alpha\,i}$, the formula (15) leads to the following forms of appearance oscillation probabilities:

$$\begin{aligned} P\left(\nu_\mu \rightarrow \nu_e\right) = &\frac{1}{49}\sin^2 x_{32} \\ &+ \frac{96}{7\cdot 49^2}\xi\left[\left(1+\frac{48}{7\cdot 49}\xi\right)\sin^2 x_{21} - \left(1-\frac{48}{7\cdot 49}\xi\right)\sin^2 x_{31}\right] \;, \end{aligned} \tag{16}$$

$$\begin{aligned} P\left(\nu_\mu \rightarrow \nu_\tau\right) = &\frac{48}{49}\sin^2 x_{32} \\ &+ \frac{96}{7\cdot 49^2}\xi\left[-\left(1-\frac{1}{7\cdot 49}\xi\right)\sin^2 x_{21} + \left(1+\frac{1}{7\cdot 49}\xi\right)\sin^2 x_{31}\right] \;, \end{aligned} \tag{17}$$

$$P\left(\nu_e \rightarrow \nu_\tau\right) = -\frac{48}{49^2}\sin^2 x_{32} + \frac{96}{49^2}\left[\left(1+\frac{23}{7\cdot 49}\xi+\frac{1}{14}\chi\right)\sin^2 x_{21}+\left(1-\frac{23}{7\cdot 49}\xi-\frac{1}{14}\chi\right)\sin^2 x_{31}\right] \tag{18}$$

as well as of survival oscillation probabilities :

$$P\left(\nu_e \rightarrow \nu_e\right) = 1-\frac{1}{49^2}\sin^2 x_{32} - \frac{96}{49^2}\left[\left(1+\frac{72}{7\cdot 49}\xi+\frac{1}{14}\chi\right)\sin^2 x_{21}+\left(1-\frac{72}{7\cdot 49}\xi-\frac{1}{14}\chi\right)\sin^2 x_{31}\right]\ , \tag{19}$$

$$P\left(\nu_\mu \rightarrow \nu_\mu\right) = 1-\sin^2 x_{32}-\frac{96}{49^3}\xi^2\left(\sin^2 x_{21}+\sin^2 x_{31}\right)\ , \tag{20}$$

$$P\left(\nu_\tau \rightarrow \nu_\tau\right) = 1-\left(\frac{48}{49}\right)^2\sin^2 x_{32} - \frac{96}{49^2}\left[\left(1-\frac{26}{7\cdot 49}\xi+\frac{1}{14}\chi\right)\sin^2 x_{21}+\left(1+\frac{26}{7\cdot 49}\xi-\frac{1}{14}\chi\right)\sin^2 x_{31}\right]\ . \tag{21}$$

Thus, we get $P\left(\nu_e \rightarrow \nu_e\right)+P\left(\nu_e \rightarrow \nu_\mu\right)+P\left(\nu_e \rightarrow \nu_\tau\right)=1$ and two other obvious summation rules for probabilities. Among these probabilities, $P\left(\nu_\mu \rightarrow \nu_\mu\right)$ displays (in the lowest perturbative order) maximal mixing between ν_2 and ν_3.

In the lowest perturbative order,

$$x_{31}-x_{21}=x_{32}=14\left(\frac{48}{49}\xi+\chi\right)\left(1.26693|M_{12}^{(\nu)}|\,L/E\right) \tag{22}$$

due to Eqs. (8) and (14). Hence,

$$\sin^2 x_{31}=\sin^2 x_{21}+x_{32}\sin 2x_{21}+x_{32}^2\sin 2x_{21} \tag{23}$$

in experiments where $x_{32}\ll\pi/2$. When in such cases the relation (23) is inserted into the formulae (16), (17) and (20), its x_{32} and x_{32}^2 terms can be neglected in the lowest perturbative order.

Note that the mass formulae (5) imply $m_{\nu_1}^2\ll m_{\nu_2}^2\lesssim m_{\nu_3}^2$, where $m_{\nu_1}^2/m_{\nu_2,\nu_3}^2=\xi^2/49^3+O(\xi^3)$ and $m_{\nu_2}^2/m_{\nu_3}^2=1-(2/7)(48\xi/49+\chi)+O(\xi^3)$. Thus, the inequality $x_{31}\gtrsim x_{21}\gg x_{32}$ holds in all neutrino oscillation experiments (with some given L and E).

We have calculated the neutrino masses, lepton Cabibbo—Kobayashi—Maskawa matrix and neutrino oscillation probabilities also in the next to lowest perturbative order. Then, in Eqs. (5) the mass m_{ν_1} gets no quadratic correction, while m_{ν_2} and m_{ν_3} are corrected by the terms

$$\mp \frac{1}{14}\left(\frac{13\cdot 48}{49^2}\xi^2 - \frac{24}{49}\xi\chi + \frac{1}{4}\chi^2\right)|M^{(\nu)}_{12}| \,, \tag{24}$$

respectively. Among the derived oscillation formulae, Eq. (20), for instance, is extended to the form

$$\begin{aligned} P\left(\nu_\mu \to \nu_\mu\right) = 1 - &\left(1 - \frac{672}{49^3}\xi^2 + \frac{24}{49^2}\xi\chi - \frac{1}{4\cdot 49}\chi^2\right)\sin^2 x_{32} \\ &-\frac{96}{49^3}\xi^2\left(\sin^2 x_{21} + \sin^2 x_{31}\right) \\ = 1 - &\left(1 - 0.00514\,\xi^2\right)\sin^2 x_{32} - 0.000816\,\xi^2\left(\sin^2 x_{21} + \sin^2 x_{31}\right) \end{aligned} \tag{25}$$

displaying nearly maximal mixing between ν_2 and ν_3.

In the case of Super–Kamiokande atmospheric neutrino experiment [4], if $\nu_\mu \to \nu_\tau$ oscillations are responsible for the observed deficit of atmospheric ν_μ's, we have $x_{\rm atm} = x_{32} \ll x_{21} \lesssim x_{31}$, what implies that $\sin^2 x_{21} = \sin^2 x_{31} = 1/2$ due to averaging over many oscillation lengths. Then, Eq. (25) leads to the following effective two–flavor oscillation formula:

$$P\left(\nu_\mu \to \nu_\mu\right) = 1 - \left(1 - 0.00350\,\xi^2\right)\sin^2 x_{32} \,, \tag{26}$$

if we assume in Eq. (25) that $0.000816\xi^2 = 0.000816\xi^2(2\sin^2 x_{32})$ effectively. Identifying the estimation (26) with the two–flavor formula fitted in the Super–Kamiokande experiment, we obtain the limits

$$\begin{aligned} 1 - 0.00350\,\xi^2 &\equiv \sin^2 2\theta_{\rm atm} \sim 0.82 \;\text{ to }\; 1 \,, \\ \Delta m^2_{32} &\equiv \Delta m^2_{\rm atm} \sim (0.5 \;\text{ to }\; 6)\times 10^{-3}\,{\rm eV}^2 \,. \end{aligned} \tag{27}$$

Hence, $\xi \sim 7.17$ to 0 and

$$\begin{aligned} \frac{\mu^{(\nu)}}{\alpha^{(\nu)}} &\equiv 0.00334\xi \sim 0.0239 \;\text{ to }\; 0 \,, \\ \alpha^{(\nu)}\mu^{(\nu)} &\equiv 0.483\Delta m^2_{32} \sim (0.241 \;\text{ to }\; 2.90)\times 10^{-4}\,{\rm eV}^2 \,, \end{aligned} \tag{28}$$

where Eqs. (6) and (8) are used. For instance, with $\sin^2 2\theta_{\rm atm} \sim 0.999$ and $\Delta m^2_{\rm atm} \sim 5\times 10^{-3}\,{\rm eV}^2$, we get $\xi \sim 0.535$ and

$$\frac{\mu^{(\nu)}}{\alpha^{(\nu)}} \sim 0.00178 \quad , \quad \alpha^{(\nu)}\mu^{(\nu)} \sim 2.41 \times 10^{-4}\,\mathrm{eV}^2 \;, \tag{29}$$

what gives the estimation

$$\alpha^{(\nu)} \sim 0.368\,\mathrm{eV} \quad , \quad \mu^{(\nu)} \sim 6.55 \times 10^{-4}\,\mathrm{eV} \;. \tag{30}$$

Note that $\xi < 1$ for $\sin^2 2\theta_{\mathrm{atm}} > 0.9965$. As was already mentioned, our actual perturbative parameters are not ξ and χ, but rather $\xi/7$ and $\chi/7 = 0.0594\xi/7$.

Having estimated $\alpha^{(\nu)}$ and $\mu^{(\nu)}$, we can calculate neutrino masses from Eqs. (5) with (6) and (7). Making use of the values (30) (valid for $\sin^2 2\theta_{\mathrm{atm}} \sim 0.999$ and $\Delta m^2_{\mathrm{atm}} \sim 5 \times 10^{-3}\,\mathrm{eV}^2$), we obtain

$$m_{\nu_1} \sim 2.76 \times 10^{-4}\,\mathrm{eV} \quad , \quad m_{\nu_2} \sim -1.71 \times 10^{-1}\,\mathrm{eV} \quad , \quad m_{\nu_3} \sim 1.85 \times 10^{-1}\,\mathrm{eV}. \tag{31}$$

Because of the smallness of these masses, the neutrinos ν_1, ν_2, ν_3 are not likely to be responsible for the entire hot dark matter.

In the case of solar neutrino experiments, all three popular fits [5] of the observed deficit of solar ν_e's to an effective two–flavor oscillation formula require $\Delta m^2_{\mathrm{sol}} \ll \Delta m^2_{\mathrm{atm}}$ what implies $\Delta m^2_{\mathrm{sol}} \ll \Delta m^2_{32} \ll \Delta m^2_{21} \lesssim \Delta m^2_{31}$, if $\nu_\mu \to \nu_\tau$ oscillations are responsible for the deficit of atmospheric ν_μ's. Then, $x_{\mathrm{sol}} \ll x_{32} \ll x_{21} \lesssim x_{31}$, giving $\sin^2 x_{32} = \sin^2 x_{21} = \sin^2 x_{31} = 1/2$ due to averaging over many oscillation lengths. In such a case, Eq. (19) leads to

$$P\left(\nu_e \to \nu_e\right) = 1 - \frac{193}{2 \cdot 49^2} = 1 - 0.0402 = 0.960 \;, \tag{32}$$

predicting only a 4% deficit of solar ν_e's, much too small to explain solar neutrino observations.

An intriguing situation arises in the case of formula (16) for $P\left(\nu_\mu \to \nu_e\right)$, if $\nu_\mu \to \nu_\tau$ oscillations really cause the bulk of deficit of atmospheric ν_μ's. Then, for a new $x_{\mathrm{new}} = x_{32} \ll x_{21} \lesssim x_{31}$ (with some new L and E) we may have $\sin^2 x_{21} = \sin^2 x_{31} = 1/2$ due to averaging over many oscillation lengths and so, infer from Eq. (16) that

$$P\left(\nu_\mu \to \nu_e\right) = \frac{1}{49}\sin^2 x_{32} + \frac{2 \cdot 48^2}{49^4}\xi^2 \sim 0.0204 \sin^2 x_{32} + 2.29 \times 10^{-4} \;, \tag{33}$$

where $\xi^2 \sim 0.286$ (what is valid for $\sin^2 2\theta_{\rm atm} \sim 0.999$ and $\Delta m^2_{\rm atm} \sim 5 \times 10^{-3}\,{\rm eV}^2$). Such a predicted oscillation amplitude $\sin^2 2\theta_{\rm new} \sim 0.02$ would lie in the range of $\sin^2 2\theta_{\rm LSND}$ estimated in the positive (though still requiring confirmation) LSND accelerator experiment on $\nu_\mu \to \nu_e$ oscillations [6]. However, the lower limit $\Delta m^2_{\rm LSND} \gtrsim 0.1\,{\rm eV}^2$ reported by this experiment is by one order of magnitude larger than the Super–Kamiokande upper limit $\Delta m^2_{32} \lesssim 0.01\,{\rm eV}^2$. On the other hand, the small predicted oscillation amplitude $\sin^2 2\theta_{\rm new} \sim 0.02$ would not be in conflict with the negative result of the CHOOZ long–baseline reactor experiment on $\bar{\nu}_e \to \bar{\nu}_\mu$ oscillations [7].

In conclusion, our explicit model of lepton texture displays a number of important features. *(i)* It correlates correctly (with high precision) the tauon mass with electron and muon masses. *(ii)* It predicts (without parameters) the maximal mixing between muon and tauon neutrinos in the limit $\mu^{(\nu)} \to 0$, consistent with the observed deficit of atmospheric ν_μ's. *(iii)* It fails to explain the observed deficit of solar ν_e's. *(iv)* It predicts new $\nu_\mu \to \nu_e$ oscillations with the amplitude consistent with LSND experiment, but with a phase corresponding to the mass squared difference at least one order of magnitude smaller.

In the framework of our model, the point *(iii)* may suggest that in Nature there exists (at least) one sort, $\nu_s^{(e)}$, of sterile neutrinos (blind to the Standard Model interactions), responsible for the observed deficit of solar ν_e's through $\nu_e \to \nu_s^{(e)}$ oscillations dominating the survival probability $P(\nu_e \to \nu_e) \simeq 1 - P(\nu_e \to \nu_s^{(e)})$ [8]. In an extreme version of this picture, it might even happen that in Nature there would be two sorts, $\nu_s^{(e)}$ and $\nu_s^{(\mu)}$, of sterile neutrinos, where $\nu_s^{(\mu)}$ would replace ν_τ in explaining the observed deficit of atmospheric ν_μ's by means of $\nu_\mu \to \nu_s^{(\mu)}$ oscillations that should dominate the survival probability $P(\nu_\mu \to \nu_\mu) \simeq 1 - P(\nu_\mu \to \nu_s^{(\mu)})$ [9]. In this case, the constant $\alpha^{(\nu)}$ for active neutrinos might be even zero (however, very small $\alpha^{(\nu)}$ would be still allowed). Such a model is discussed in Sections 5 and 6.

For the author of the present paper the idea of existence of two sorts of sterile neutrinos is fairly appealing, since two such spin–1/2 fermions, blind to all Standard Model interactions, do follow (besides three standard families of active leptons and quarks) [8] from the argument *(i)* mentioned in Introduction, based on the Kähler–like generalized Dirac equations. Note in addition that the $\nu_e \to \nu_s^{(e)}$ and $\nu_\mu \to \nu_s^{(\mu)}$ oscillations caused

by appropriate mixings should be a natural consequence of the spontaneous breaking of electroweak $SU(2) \times U(1)$ symmetry.

In Section 7, a possibility is considered that two extra neutrino mass states, whose existence is implied by two sterile neutrinos $\nu_s^{(e)}$ and $\nu_s^{(\mu)}$, cause in the Standard Model framework some tiny neutrino instability and related damping of ν_e and ν_μ oscillations.

4. Perspectives for unification with quarks

In this Section, we try to apply to quarks the form of mass matrix which was worked out above for leptons. To this end, we conjecture for three generations of up quarks u , c , t and down quarks d , s , b the mass matrices $\left(M_{\alpha\beta}^{(u)}\right)$ and $\left(M_{\alpha\beta}^{(d)}\right)$, respectively, essentially of the form (1), where the label $f = u\,,\, d$ denotes now up and down quarks. The only modification introduced is a new real constant $C^{(f)}$ added to $\varepsilon^{(f)}$ in the element $M_{33}^{(f)}$ which now reads

$$M_{33}^{(f)} = \frac{24\mu^{(f)}}{25 \cdot 29}\left(624 + \varepsilon^{(f)} + C^{(f)}\right) \,. \tag{34}$$

Since for quarks the mass scales $\mu^{(u)}$ and $\mu^{(d)}$ are expected to be even more important than the scale $\mu^{(e)}$ for charged leptons, we assume that the off–diagonal elements of mass matrices $\left(M_{\alpha\beta}^{(u)}\right)$ and $\left(M_{\alpha\beta}^{(d)}\right)$ can be considered as a small perturbation of their diagonal terms. Then, in the lowest perturbative order, we obtain the following mass formulae

$$\begin{aligned}
m_{u,d} &= \frac{\mu^{(u,d)}}{29}\varepsilon^{(u,d)} - A^{(u,d)}\left(\frac{\alpha^{(u,d)}}{\mu^{(u,d)}}\right)^2 \,, \\
m_{c,s} &= \frac{\mu^{(u,d)}}{29}\frac{4}{9}\left(80 + \varepsilon^{(u,d)}\right) + \left(A^{(u,d)} - B^{(u,d)}\right)\left(\frac{\alpha^{(u,d)}}{\mu^{(u,d)}}\right)^2 \,, \\
m_{t,b} &= \frac{\mu^{(u,d)}}{29}\frac{24}{25}\left(624 + \varepsilon^{(u,d)} + C^{(u,d)}\right) + B^{(u,d)}\left(\frac{\alpha^{(u,d)}}{\mu^{(u,d)}}\right)^2 \,,
\end{aligned} \tag{35}$$

where

$$A^{(u,d)} = \frac{\mu^{(u,d)}}{29}\,\frac{36}{320 - 5\varepsilon^{(u,d)}} \;\;,\;\; B^{(u,d)} = \frac{\mu^{(u,d)}}{29}\,\frac{10800}{31696 + 54C^{(u,d)} + 29\varepsilon^{(u,d)}} \,. \tag{36}$$

In Eqs. (35), the relative smallness of perturbating terms is more pronounced due to extra factors. In our discussion, we will take for experimental quark masses the arithmetic means of their lower and upper limits quoted in the Review of Particle Physics [3] *i.e.*,

$$m_u = 3.3\,\mathrm{MeV}\;,\;\; m_c = 1.3\,\mathrm{GeV}\;,\;\; m_t = 174\,\mathrm{GeV} \tag{37}$$

and

$$m_d = 6\,\mathrm{MeV}\;,\;\; m_s = 120\,\mathrm{MeV}\;,\;\; m_b = 4.3\,\mathrm{GeV}\;. \tag{38}$$

Eliminating from the unperturbed terms in Eqs. (35) the constants $\mu^{(u,d)}$ and $\varepsilon^{(u,d)}$, we derive the correlating formulae being counterparts of Eqs. (2) for charged leptons:

$$\begin{aligned} m_{t,b} &= \frac{6}{125}\left(351 m_{c,s} - 136 m_{u,d}\right) + \frac{\mu^{(u,d)}}{29}\frac{24}{25}C^{(u,d)} \\ &\quad -\frac{1}{125}\left(2922 A^{(u,d)} - 2231 B^{(u,d)}\right)\left(\frac{\alpha^{(u,d)}}{\mu^{(u,d)}}\right)^2 \;, \\ \mu^{(u,d)} &= \frac{29}{320}\left(9 m_{c,s} - 4 m_{u,d}\right) - \frac{29}{320}\left(5A^{(u,d)} - 9B^{(u,d)}\right)\left(\frac{\alpha^{(u,d)}}{\mu^{(u,d)}}\right)^2 \;, \\ \varepsilon^{(u,d)} &= \frac{29 m_{u,d}}{\mu^{(u,d)}} + \frac{29}{\mu^{(u,d)}}A^{(u,d)}\left(\frac{\alpha^{(u,d)}}{\mu^{(u,d)}}\right)^2 \;. \end{aligned} \tag{39}$$

The unperturbed parts of these relations are:

$$\begin{aligned} \overset{\circ}{m}_{t,b} &= \frac{6}{125}\left(351 m_{c,s} - 136 m_{u,d}\right) + \frac{\overset{\circ}{\mu}{}^{(u,d)}}{29}\frac{24}{25}\overset{\circ}{C}{}^{(u,d)} \\ &= \left\{\begin{array}{c} 21.9 \\ 1.98 \end{array}\right\}\mathrm{GeV} + \frac{\overset{\circ}{\mu}{}^{(u,d)}}{29}\frac{24}{25}\overset{\circ}{C}{}^{(u,d)} \;, \\ \overset{\circ}{\mu}{}^{(u,d)} &= \frac{29}{320}\left(9 m_{c,s} - 4 m_{u,d}\right) = \left\{\begin{array}{c} 1060 \\ 95.7 \end{array}\right\}\mathrm{MeV}\;, \\ \overset{\circ}{\varepsilon}{}^{(u,d)} &= \frac{29 m_{u,d}}{\overset{\circ}{\mu}{}^{(u,d)}} = \left\{\begin{array}{c} 0.0904 \\ 1.82 \end{array}\right\}\;. \end{aligned} \tag{40}$$

In the spirit of our perturbative approach, the "coupling" constant $\alpha^{(u,d)}$ can be put zero in all perturbing terms in Eqs. (35) and (39), except for $\alpha^{(u,d)\,2}$ in the numerator of the factor $(\alpha^{(u,d)}/\mu^{(u,d)})^2$ that now becomes $(\alpha^{(u,d)}/\overset{\circ}{\mu}{}^{(u,d)})^2$. Then, $A^{(u,d)}$ and $B^{(u,d)}$ are replaced by

$$\overset{\circ}{A}{}^{(u,d)} = \frac{\overset{\circ}{\mu}{}^{(u,d)}}{29}\frac{36}{320 - 5\,\overset{\circ}{\varepsilon}{}^{(u,d)}}\;\;,\;\; \overset{\circ}{B}{}^{(u,d)} = \frac{\overset{\circ}{\mu}{}^{(u,d)}}{29}\frac{10800}{31696 + 54\,\overset{\circ}{C}{}^{(u,d)} + 29\,\overset{\circ}{\varepsilon}{}^{(u,d)}}\;. \tag{41}$$

Note that the first Eq. (35) can be rewritten identically as $m_{u,d} = \stackrel{\circ}{\mu}{}^{(u,d)} \stackrel{\circ}{\varepsilon}{}^{(u,d)}/29$ according to the third Eq. (40).

We shall be able to return to the discussion of quark masses after the estimation of constants $\alpha^{(u)}$ and $\alpha^{(d)}$ is made. Then, we shall determine the parameters $C^{(u)}$ and $C^{(d)}$ (as well as their unperturbed parts $\stackrel{\circ}{C}{}^{(u)}$ and $\stackrel{\circ}{C}{}^{(d)}$) playing here an essential role in providing large values for m_t and m_b.

At present, we find the unitary matrices $\left(U^{(u,d)}_{\alpha\beta}\right)$ that diagonalize the mass matrices $\left(M^{(u,d)}_{\alpha\beta}\right)$ according to the relations $U^{(u,d)\,\dagger} M^{(u,d)} U^{(u,d)} = \mathrm{diag}(m_{u,d}\,,\, m_{c,s}\,,\, m_{t,b})$. In the lowest perturbative order, the result has the form (4) with the necessary replacement of labels:

$$(e) \rightarrow (u) \;\; \mathrm{or} \;\; (d)\;,\; \mu \rightarrow c \;\; \mathrm{or} \;\; s\;,\; \tau \rightarrow t \;\; \mathrm{or} \;\; b\;, \tag{42}$$

respectively.

Then, the elements $V_{\alpha\beta}$ of the Cabibbo—Kobayashi—Maskawa matrix $V = U^{(u)\,\dagger} U^{(d)}$ can be calculated with the use of Eqs. (42) in the lowest perturbative order. Six resulting off–diagonal elements are:

$$\begin{aligned}
V_{us} &= -V^*_{cd} = \frac{2}{29}\left(\frac{\alpha^{(d)}}{m_s} e^{i\varphi^{(d)}} - \frac{\alpha^{(u)}}{m_c} e^{i\varphi^{(u)}}\right)\;, \\
V_{cb} &= -V^*_{ts} = \frac{8\sqrt{3}}{29}\left(\frac{\alpha^{(d)}}{m_b} e^{i\varphi^{(d)}} - \frac{\alpha^{(u)}}{m_t} e^{i\varphi^{(u)}}\right) \simeq \frac{8\sqrt{3}}{29}\frac{\alpha^{(d)}}{m_b} e^{i\varphi^{(d)}}\;, \\
V_{ub} &\simeq -\frac{16\sqrt{3}}{841}\frac{\alpha^{(u)}\alpha^{(d)}}{m_c m_b} e^{i(\varphi^{(u)}+\varphi^{(d)})}\;, \\
V_{td} &\simeq \frac{16\sqrt{3}}{841}\frac{\alpha^{(d)\,2}}{m_s m_b}\, e^{-2i\varphi^{(d)}}\;,
\end{aligned} \tag{43}$$

where the indicated approximate steps were made due to the inequality $m_t \gg m_b$ and/or under the assumption that $\alpha^{(u)}/m_c \gg \alpha^{(d)}/m_b$ [*cf.* the conjecture (46)]. All three diagonal elements are real and positive in a good approximation:

$$V_{ud} \simeq 1 - \frac{1}{2}|V_{us}|^2\;,\; V_{cs} \simeq 1 - \frac{1}{2}|V_{us}|^2 - \frac{1}{2}|V_{cb}|^2\;,\; V_{tb} \simeq 1 - \frac{1}{2}|V_{cb}|^2\;. \tag{44}$$

In fact, in the lowest perturbative order,

$$\arg V_{ud} \simeq \frac{4}{841} \frac{\alpha^{(u)}\alpha^{(d)}}{m_c m_s} \sin\left(\varphi^{(u)} - \varphi^{(d)}\right) \frac{180^\circ}{\pi} \simeq -\arg V_{cs} \;,\; \arg V_{tb} \simeq 0 \;, \tag{45}$$

what gives $\arg V_{ud} = 0.88^\circ = -\arg V_{cs}$, if the values (46), (49) and (52) are used.

Taking as an input the experimental value $|V_{cb}| = 0.0395 \pm 0.0017$ [3], we estimate from the second Eq. (43) that

$$\alpha^{(d)} \simeq \frac{29}{8\sqrt{3}} m_b \, |V_{cb}| = (355 \pm 15) \text{ MeV} \;, \tag{46}$$

where $m_b = 4.3$ GeV. In order to estimate also $\alpha^{(u)}$, we will tentatively conjecture the approximate proportion

$$\alpha^{(u)} : \alpha^{(d)} \simeq Q^{(u)\,2} : Q^{(d)\,2} = 4 \tag{47}$$

to hold, where $Q^{(u)} = 2/3$ and $Q^{(d)} = -1/3$ are quark electric charges. Note that in the case of leptons we had $\alpha^{(\nu)} : \alpha^{(e)} = 0.37 : (\sqrt{180} \times 10^6) = 2.8 \times 10^{-8}$ for the central value of $\alpha^{(e)}$ [*cf.* Eqs. (3) and (30)], what is consistent with the analogical approximate proportion

$$\alpha^{(\nu)} : \alpha^{(e)} \simeq Q^{(\nu)\,2} : Q^{(e)\,2} = 0 \;, \tag{48}$$

where $Q^{(\nu)} = 0$ and $Q^{(e)} = -1$ are lepton electric charges. Under the conjecture (47):

$$\alpha^{(u)} \simeq (1420 \pm 60)\,\text{MeV} \;. \tag{49}$$

In this case, from the second and third Eq. (43) we obtain the prediction

$$|V_{ub}|/|V_{cb}| \simeq \frac{2}{29} \frac{\alpha^{(u)}}{m_c} \simeq 0.0753 \pm 0.0032 \;, \tag{50}$$

where $m_c = 1.3$ GeV. This is consistent with the experimental figure $|V_{ub}|/|V_{cb}| = 0.08 \pm 0.02$ [3].

Now, with the experimental value $|V_{us}| = 0.2196 \pm 0.0023$ [3] as another input, we can calculate from the first Eq. (43) the phase difference $\varphi^{(u)} - \varphi^{(d)}$. In fact, taking the absolute value of this equation, we get

$$\cos\left(\varphi^{(u)}-\varphi^{(d)}\right)=\frac{1}{8}\frac{m_c}{m_s}\left[1+16\left(\frac{m_s}{m_c}\right)^2-\frac{841}{4}\left(\frac{m_c}{\alpha^{(d)}}\right)^2|V_{us}|^2\right]=-0.0301 \tag{51}$$

with m_c = 1.3 GeV and m_s = 120 MeV, if the proportion (47) is taken into account. Here, the central values of $\alpha^{(d)}$ and $|V_{us}|$ were used. Hence,

$$\varphi^{(u)}-\varphi^{(d)}=91.7^\circ=-88.3^\circ+180^\circ \tag{52}$$

so, this phase difference turns out to be near 90°. Then, calculating the argument of the first Eq. (43), we infer that

$$\tan\left(\arg V_{us}-\varphi^{(d)}\right)=-4\,\frac{m_s}{m_c}\,\frac{\sin\left(\varphi^{(u)}-\varphi^{(d)}\right)}{1-4(m_s/m_c)\cos\left(\varphi^{(u)}-\varphi^{(d)}\right)}=-0.365\;, \tag{53}$$

what gives

$$\arg V_{us}=-20.1^\circ+\varphi^{(d)}\;. \tag{54}$$

The results (52) and (54) together with the formula (43) enable us to evaluate the rephasing–invariant CP–violating phases

$$\arg(V^*_{us}V^*_{cb}V_{ub})=20.1^\circ-88.3^\circ=-68.2^\circ \tag{55}$$

and

$$\arg(V^*_{cd}V^*_{ts}V_{td})=-20.1^\circ\;, \tag{56}$$

which turn out to be near to -70° and -20°, respectively (they are invariant under quark rephasing equal for up and down quarks of the same generation). Note that the sum of arguments (55) and (56) is always equal to $\varphi^{(u)}-\varphi^{(d)}-180^\circ$. Carrying out quark rephasing (equal for up and down quarks of the same generation), where

$$\arg V_{us}\to 0\;,\;\arg V_{cb}\to 0\;,\;\arg V_{cd}\to 180^\circ\;,\;\arg V_{ts}\to 180^\circ \tag{57}$$

and $\arg V_{ud}$, $\arg V_{cs}$, $\arg V_{tb}$ remain unchanged, we conclude from Eqs. (55) and (56) that

$$\arg V_{ub} \to -68.2^\circ \ , \ \arg V_{td} \to -20.1^\circ \ . \tag{58}$$

The sum of arguments (58) after rephasing (57) is always equal to $\varphi^{(u)} - \varphi^{(d)} - 180^\circ$.

Thus, in this quark phasing, we predict the following Cabibbo—Kobayashi—Maskawa matrix:

$$(V_{\alpha\beta}) = \left(\begin{array}{ccc} 0.976 & 0.220 & 0.00297\, e^{-i\,68.2^\circ} \\ -0.220 & 0.975 & 0.0395 \\ 0.00805\, e^{-i\,20.1^\circ} & -0.0395 & 0.999 \end{array} \right) . \tag{59}$$

Here, only $|V_{us}|$ and $|V_{cb}|$ [and quark masses $m_s\,,\ m_c\,,\ m_b$ consistent with the mass matrices $\left(M^{(u)}_{\alpha\beta}\right)$ and $\left(M^{(d)}_{\alpha\beta}\right)$] are our inputs, while all other matrix elements $V_{\alpha\beta}$, partly induced by unitarity, are evaluated from the relations derived in this Section from the Hermitian mass matrices $\left(M^{(u)}_{\alpha\beta}\right)$ and $\left(M^{(d)}_{\alpha\beta}\right)$ [and the conjectured proportion (47)]. The independent predictions are $|V_{ub}|$ and argV_{ub}. In Eq. (59), the small phases arising from Eqs. (45), $\arg V_{ud} = 0.9^\circ$ and $\arg V_{cs} = -0.9^\circ$, are neglected (here, arg $(V_{ud} V_{cs} V_{tb}) = 0$).

The above prediction of $V_{\alpha\beta}$ implies the following values of Wolfenstein parameters [3]:

$$\lambda = 0.2196 \ \ , \ \ A = 0.819 \ \ , \ \ \rho = 0.127 \ \ , \ \ \eta = 0.319 \tag{60}$$

and of unitary–triangle angles:

$$\gamma = \arctan \frac{\eta}{\rho} = -\arg V_{ub} = 68.2^\circ \ \ , \ \ \beta = \arctan \frac{\eta}{1-\rho} = -\arg V_{td} = 20.1^\circ \ . \tag{61}$$

The predicted large value of γ follows the present experimental tendency.

If instead of the central value $|V_{us}| = 0.2196$ we take as the input the range $|V_{us}| = 0.2173$ to 0.2219, we obtain from Eq. (51) $\varphi^{(u)} - \varphi^{(d)} = 89.8^\circ$ to 93.6° (with $|V_{cb}| = 0.0395$ giving $\alpha^{(d)} = 355$ MeV), what implies through Eq. (53) that arg $V_{us} - \varphi^{(d)} = -20.3^\circ$ to -19.8°. Then, after rephasing (57), arg$V_{ub} = -69.9^\circ$ to -66.6° and $\arg V_{td} = -20.3^\circ$ to -19.8°. In this case, the Wolfenstein parameters are $\lambda = 0.2173$ to 0.2219, $A = 0.837$ to 0.802, $\rho = 0.119$ to 0.135 and $\eta = 0.325$ to 0.312 (here, $\lambda\sqrt{\rho^2+\eta^2} = |V_{ub}|/|V_{cb}| = 0.0753$ is fixed). Thus, $\gamma = -\arg V_{ub} = 69.9^\circ$ to 66.6° and $\beta = -\arg V_{td} = 20.3^\circ$ to 19.8°.

In contrast, if the central value $|V_{cd}| = 0.0395$ (giving $\alpha^{(d)} = 355$ MeV) is replaced by the input of the range $V_{cd} = 0.0378$ to 0.0412 (corresponding to $\alpha^{(d)} = 340$ to 370 MeV), we calculate from Eq. (51) that $\varphi^{(u)} - \varphi^{(d)} = 97.3^\circ$ to 84.9° (with $|V_{us}| = 0.2196$), what leads to $\arg V_{us} - \varphi^{(d)} = -19.3^\circ$ to -20.9°. Hence, after rephasing (57), $\arg V_{ub} = -63.4^\circ$ to -74.6° and $\arg V_{td} = -19.3^\circ$ to -20.9°. In this case, the Wolfenstein parameters take the values $\lambda = 0.2196$, $A = 0.784$ to 0.854, $\rho = 0.149$ to 0.0951 and $\eta = 0.298$ to 0.345. Thus, $\gamma = -\arg V_{ub} = 63.4^\circ$ to 74.6° and $\beta = -\arg V_{td} = 19.3^\circ$ to 20.9°. Here, $|V_{ub}| = 0.00273$ to 0.00323 and $|V_{td}| = 0.00738$ to 0.00874.

Eventually, we may turn back to quark masses. From the third Eq. (35) we can evaluate

$$C^{(u,d)} = \frac{29}{\mu^{(u,d)}}\frac{25}{24} m_{t,b} - 624 - \varepsilon^{(u,d)} - \frac{29}{\mu^{(u,d)}}\frac{25}{24} B^{(u,d)} \left(\frac{\alpha^{(u,d)}}{\mu^{(u,d)}}\right)^2 , \tag{62}$$

what, in the framework of our perturbative approach, gives

$$\begin{aligned} C^{(u,d)} &= \overset{\circ}{C}{}^{(u,d)} + \frac{29}{\overset{\circ}{\mu}{}^{(u,d)}}\frac{25}{24} m_{t,b} \frac{29}{320\,\overset{\circ}{\mu}{}^{(u,d)}} \left(5\,\overset{\circ}{A}{}^{(u,d)} - 9\,\overset{\circ}{B}{}^{(u,d)}\right) \left(\frac{\alpha^{(u,d)}}{\overset{\circ}{\mu}{}^{(u,d)}}\right)^2 \\ &\quad - \frac{29}{\overset{\circ}{\mu}{}^{(u,d)}} \left(\overset{\circ}{A}{}^{(u,d)} + \overset{\circ}{B}{}^{(u,d)}\right) \left(\frac{\alpha^{(u,d)}}{\overset{\circ}{\mu}{}^{(u,d)}}\right)^2 , \end{aligned} \tag{63}$$

where

$$\overset{\circ}{C}{}^{(u,d)} = \frac{29}{\overset{\circ}{\mu}{}^{(u,d)}}\frac{25}{24} m_{t,b} - 624 - \overset{\circ}{\varepsilon}{}^{(u,d)} = \left\{ \begin{array}{c} 4339 \\ 733.2 \end{array} \right\} = \left\{ \begin{array}{c} 4340 \\ 733 \end{array} \right\} . \tag{64}$$

With the central values of $\alpha^{(u)}$ and $\alpha^{(d)}$ as estimated in Eqs. (46) and (49) we find from Eqs. (41)

$$\overset{\circ}{A}{}^{(u,d)} \left(\frac{\alpha^{(u,d)}}{\overset{\circ}{\mu}{}^{(u,d)}}\right)^2 = \left\{ \begin{array}{c} 7.39 \\ 5.26 \end{array} \right\} \text{MeV} , \; \overset{\circ}{B}{}^{(u,d)} \left(\frac{\alpha^{(u,d)}}{\overset{\circ}{\mu}{}^{(u,d)}}\right)^2 = \left\{ \begin{array}{c} 2.66 \\ 6.88 \end{array} \right\} \text{MeV} , \tag{65}$$

where

$$\frac{\overset{\circ}{\mu}{}^{(u,d)}}{29} \left(\frac{\alpha^{(u,d)}}{\overset{\circ}{\mu}{}^{(u,d)}}\right)^2 = \left\{ \begin{array}{c} 65.6 \\ 45.4 \end{array} \right\} \text{MeV} . \tag{66}$$

We calculate from Eqs. (63) with the use of values (65) that

$$C^{(u,d)} = \left\{ \begin{array}{c} 4339 + 5.25 \\ 733.2 - 49.5 \end{array} \right\} = \left\{ \begin{array}{c} 4344 \\ 683.7 \end{array} \right\} = \left\{ \begin{array}{c} 4340 \\ 684 \end{array} \right\} . \tag{67}$$

Similarly, from the second Eq. (39), making use of the values (65), we obtain

$$\mu^{(u,d)} = \left\{ \begin{array}{c} 1060 - 1.18 \\ 95.7 + 3.23 \end{array} \right\} \mathrm{MeV} = \left\{ \begin{array}{c} 1059 \\ 98.9 \end{array} \right\} \mathrm{MeV} = \left\{ \begin{array}{c} 1060 \\ 98.9 \end{array} \right\} \mathrm{MeV} . \tag{68}$$

We can easily check that, with the values (40) for $\stackrel{\circ}{\mu}{}^{(u,d)}$ and $\stackrel{\circ}{\varepsilon}{}^{(u,d)}$ and the value (64) for $\stackrel{\circ}{C}{}^{(u,d)}$ determined as above from quark masses, the unperturbed parts of mass formulae (35) reproduce correctly these masses. In fact,

$$\begin{aligned} \stackrel{\circ}{m}_{u,d} &= \frac{\stackrel{\circ}{\mu}{}^{(u,d)}}{29} \stackrel{\circ}{\varepsilon}{}^{(u,d)} = \left\{ \begin{array}{c} 3.3 \\ 6 \end{array} \right\} \mathrm{MeV} , \\ \stackrel{\circ}{m}_{c,s} &= \frac{\stackrel{\circ}{\mu}{}^{(u,d)}}{29} \frac{4}{9} \left(80 + \stackrel{\circ}{\varepsilon}{}^{(u,d)} \right) = \left\{ \begin{array}{c} 1300 \\ 120 \end{array} \right\} \mathrm{MeV} , \\ \stackrel{\circ}{m}_{t,b} &= \frac{\stackrel{\circ}{\mu}{}^{(u,d)}}{29} \frac{24}{25} \left(624 + \stackrel{\circ}{\varepsilon}{}^{(u,d)} + \stackrel{\circ}{C}{}^{(u,d)} \right) = \left\{ \begin{array}{c} 174 \\ 4.3 \end{array} \right\} \mathrm{GeV} . \end{aligned} \tag{69}$$

The same is true for the unperturbed part of the first correlating formula (39). The — here omitted — corrections to Eqs. (69), arising from all perturbing terms in the mass formulae (35) (including the corrections from $\delta\mu^{(u,d)}$, $\delta\varepsilon^{(u,d)}$ and $\delta C^{(u,d)}$), are relatively small, *viz.*

$$\delta m_{u,d} = \left\{ \begin{array}{c} 3.7 \times 10^{-3} \\ -2.0 \times 10^{-1} \end{array} \right\} \mathrm{MeV} , \; \delta m_{c,s} = \left\{ \begin{array}{c} 9.5 \\ -3.8 \end{array} \right\} \mathrm{MeV} , \; \delta m_{t,b} = \left\{ \begin{array}{c} 170 \\ -74 \end{array} \right\} \mathrm{MeV} , \tag{70}$$

respectively.

We would like to stress that, in contrast to the case of charged leptons, where m_τ has been predicted from m_e and m_μ, in the case of up and down quarks two extra parameters $C^{(u)}$ and $C^{(d)}$ appear necessarily to provide large masses m_t and m_b (much larger than m_τ). They cause that m_t (m_b) cannot be predicted from m_u and m_c (m_d and m_s), till the new parameters are quantitatively understood.

Note that a conjecture about $C^{(u)}$ and $C^{(d)}$ might lead to a prediction for quark masses and so, introduce changes in the "experimental" quark masses (37) and (38) accepted here. The same is true for a conjecture about $\varphi^{(u)}$ and $\varphi^{(d)}$.

For instance, the conjecture that the phase difference $\varphi^{(u)} - \varphi^{(d)}$ is maximal,

$$\varphi^{(u)} - \varphi^{(d)} = 90^\circ \;, \tag{71}$$

leads through the first equality in Eq. (51) to the condition

$$1 + 16\left(\frac{m_s}{m_c}\right)^2 - \frac{841}{4}\left(\frac{m_s}{\alpha^{(d)}}\right)^2 |V_{us}|^2 = 0 \tag{72}$$

predicting for s quark the mass

$$m_s = 118.7\,\mathrm{MeV} = 119\,\mathrm{MeV} \tag{73}$$

(with $\alpha^{(d)} = 355$ MeV), being only slightly lower than the value 120 MeV used previously. Here, m_c and m_b are kept equal to 1.3 and 4.3 GeV, respectively (also masses of u , d and t quarks are not changed, while $\overset{\circ}{\mu}{}^{(d)}$, $\overset{\circ}{\varepsilon}{}^{(d)}$ and $\overset{\circ}{C}{}^{(d)}$ change slightly). Then, from the first equality in Eq. (53)

$$\tan\left(\arg V_{us} - \varphi^{(d)}\right) = -4\,\frac{m_s}{m_c} = -0.365 \;\;,\;\; \arg V_{us} = -20.1^\circ + \varphi^{(d)} \;. \tag{74}$$

After rephasing (57), this gives $\arg V_{ub} + \arg V_{td} = \varphi^{(u)} - \varphi^{(d)} - 180^\circ = -90^\circ$, where

$$\arg V_{ub} = -69.9^\circ \;\;,\;\; \arg V_{td} = -20.1^\circ \tag{75}$$

i.e., practically -70° and -20°. All $|V_{\alpha\beta}|$ remain unchanged (with our inputs of $|V_{us}| = 0.2196$ and $|V_{cb}| = 0.0395$), except for $|V_{td}|$ which changes slightly, becoming

$$|V_{td}| = 0.00814 \;. \tag{76}$$

Thus, in the Cabibbo—Kobayashi—Maskawa matrix predicted in Eq. (59), only $|V_{td}|$ and the phases (75) show some changes. The Wolfenstein parameters are

$$\rho = 0.118 \;\;,\;\; \eta = 0.322 \tag{77}$$

and λ and A unchanged (here, the sum $\rho^2 + \eta^2 = 0.118$ is also unchanged). Hence, $\gamma + \beta = 90^\circ$ and $\alpha = 180^\circ - \gamma - \beta = 90^\circ$, where

$$\gamma = \arctan\frac{\eta}{\rho} = -\arg V_{ub} = 69.9^\circ \ \ , \ \ \beta = \arctan\frac{\eta}{1-\rho} = -\arg V_{td} = 20.1^\circ \ . \tag{78}$$

So, in the case of conjecture (71), the new restrictive relation

$$\frac{\eta}{\rho} = \frac{1-\rho}{\eta} \ \ , \ \ \rho^2+\eta^2 = \rho \tag{79}$$

holds, implying the prediction

$$|V_{td}|/|V_{ub}| = \sqrt{\frac{(1-\rho)^2+\eta^2}{\rho^2+\eta^2}} = \frac{\eta}{\rho} = 2.74 \ , \tag{80}$$

due to the definition of ρ and η from V_{ub} and V_{td}. It is in agreement with our figures for $|V_{td}|$ and $|V_{ub}|$. Then, the new relationship

$$\frac{1}{4}\frac{m_c}{m_s} = \frac{\alpha^{(d)} m_c}{\alpha^{(u)} m_s} = \frac{\eta}{\rho} \tag{81}$$

follows for quark masses m_c, m_s and Wolfenstein parameters ρ, η, in consequence of Eqs. (43) and the conjectured proportion (47). Both its sides are really equal for our values of m_c, m_s and ρ, η.

Thus, summarizing, we cannot predict quark masses without an *additional* knowledge or conjecture about the constants $\mu^{(u,d)}$, $\varepsilon^{(u,d)}$, $C^{(u,d)}$, $\alpha^{(u,d)}$ and $\varphi^{(u,d)}$ (in particular, the conjecture (71) predicting m_s may be natural). However, we always describe them correctly. If we describe them *jointly* with quark mixing parameters, we obtain two independent predictions of $|V_{ub}|$ and $\gamma = -\arg V_{ub}$: the whole Cabibbo—Kobayashi—Maskawa matrix is calculated from the inputs of $|V_{us}|$ and of $|V_{ub}|$ [and of quark masses m_s, m_c and m_b consistent with the mass matrices $\left(M^{(u)}_{\alpha\beta}\right)$ and $\left(M^{(d)}_{\alpha\beta}\right)$].

Concluding this Section, we can claim that our leptonic form of mass matrix works also in a promising way for up and down quarks. But, it turns out that, in the framework of the leptonic form of mass matrix, the heaviest quarks, t and b, require an additional mechanism in order to produce the bulk of their masses (here, it is represented by the large constants $C^{(u)}$ and $C^{(d)}$). Such a mechanism, however, intervenes into the process of quark mixing only through quark masses (practically m_t and m_b) and so, it does not modify for quarks the leptonic form of mixing mechanism.

5. A model of texture with two sterile neutrinos

Assume that there are two sorts, $\nu_s^{(e)}$ and $\nu_s^{(\mu)}$, of sterile neutrinos (blind to all Standard Model interactions and so, interacting only gravitationally). Conjecture that their mixings with two active neutrinos ν_e and ν_μ, respectively, dominate all neutrino mixings. Thus, five flavor neutrino fields, $\nu_\alpha = \nu_e, \nu_\mu, \nu_\tau, \nu_s^{(e)}, \nu_s^{(\mu)}$, exist in this texture and mix according to a neutrino mass matrix $M^{(\nu)}$. This can be assumed consistently in the following 5×5 form:

$$M^{(\nu)} = \left(M^{(\nu)}_{\alpha\beta}\right) = \begin{pmatrix} M^{(\nu)}_{11} & 0 & 0 & M^{(\nu)}_{14} & 0 \\ 0 & M^{(\nu)}_{22} & 0 & 0 & M^{(\nu)}_{25} \\ 0 & 0 & M^{(\nu)}_{33} & 0 & 0 \\ M^{(\nu)}_{41} & 0 & 0 & 0 & 0 \\ 0 & M^{(\nu)}_{52} & 0 & 0 & 0 \end{pmatrix} \tag{82}$$

with $M^{(\nu)}_{\alpha\beta} = M^{(\nu)*}_{\beta\alpha}$, $M^{(\nu)}_{\alpha\alpha} = |M^{(\nu)}_{\alpha\alpha}|$ and $M^{(\nu)}_{\alpha\beta} = |M^{(\nu)}_{\alpha\beta}| \exp\left(i\varphi^{(\nu)}\right)$ for $\alpha < \beta$, where the diagonal elements $M^{(\nu)}_{11}$, $M^{(\nu)}_{22}$ and $M^{(\nu)}_{33}$ are given in terms of $\mu^{(\nu)}$ and $\varepsilon^{(\nu)}$ as in Eq. (1) (with $f = \nu$). Here, we put $M^{(\nu)}_{44} = 0 = M^{(\nu)}_{55}$ and even $M^{(\nu)}_{12} = 0 = M^{(\nu)}_{23}$, the latter implying $\alpha^{(\nu)} = 0$ due to Eq. (1) (with $f = \nu$). With such a specific ansatz as (82), all neutrino mixings are caused by the existence of sterile neutrinos responsible for the off-diagonal matrix elements $M^{(\nu)}_{14}$ and $M^{(\nu)}_{25}$.

It is important to notice that, according to the useful formula for electric charge, $Q = I_3^L + Y/2$ with $Y/2 = I_3^R + (B-L)/2$, sterile neutrinos can carry no lepton number, $L = 0$. This may be a reason for $M^{(\nu)}_{44} = 0 = M^{(\nu)}_{55}$. On the other hand, the off-diagonal matrix elements $M^{(\nu)}_{14}$ and $M^{(\nu)}_{25}$, if nonzero, violate the lepton number conservation.

The mass matrix of the form (82) leads to the following masses corresponding to five mass neutrino fields $\nu_i = \nu_1\,, \nu_2\,, \nu_3\,, \nu_4\,, \nu_5$:

$$\begin{aligned} m_{\nu_1,\nu_4} &= \frac{M^{(\nu)}_{11}}{2} \pm \sqrt{\left(\frac{M^{(\nu)}_{11}}{2}\right)^2 + |M^{(\nu)}_{14}|^2}\;, \\ m_{\nu_3} &= M^{(\nu)}_{33}\;, \\ m_{\nu_2,\nu_5} &= \frac{M^{(\nu)}_{22}}{2} \pm \sqrt{\left(\frac{M^{(\nu)}_{22}}{2}\right)^2 + |M^{(\nu)}_{25}|^2}\;. \end{aligned} \tag{83}$$

Note that in Eq. (82) we used for simplicity $\alpha = 1\,,\,2\,,\,3\,,\,4\,,\,5$, which convention, if used properly, does not introduce any serious confusion with $i = 1\,,\,2\,,\,3\,,\,4\,,\,5$.

The corresponding 5×5 unitary matrix $U^{(\nu)}$, diagonalizing the neutrino mass matrix (82) according to the relation $U^{(\nu)\,\dagger}\, M^{(\nu)}\, U^{(\nu)} = \mathrm{diag}(m_{\nu_1}\,,\, m_{\nu_2}\,,\, m_{\nu_3}\,,\, m_{\nu_4}\,,\, m_{\nu_5})$, takes the form

$$U^{(\nu)} = \left(U^{(\nu)}_{\alpha i}\right) = \left(\begin{array}{ccccc} \frac{1}{\sqrt{1+X^2}} & 0 & 0 & -\frac{X}{\sqrt{1+X^2}}e^{i\varphi^{(\nu)}} & 0 \\ 0 & \frac{1}{\sqrt{1+Y^2}} & 0 & 0 & -\frac{Y}{\sqrt{1+Y^2}}e^{i\varphi^{(\nu)}} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{X}{\sqrt{1+X^2}}e^{-i\varphi^{(\nu)}} & 0 & 0 & \frac{1}{\sqrt{1+X^2}} & 0 \\ 0 & \frac{Y}{\sqrt{1+Y^2}}e^{-i\varphi^{(\nu)}} & 0 & 0 & \frac{1}{\sqrt{1+Y^2}} \end{array}\right)\,, \tag{84}$$

where

$$\begin{aligned} X &= \frac{m_{\nu_1} - M^{(\nu)}_{11}}{|M^{(\nu)}_{14}|} = -\frac{M^{(\nu)}_{11}}{2|M^{(\nu)}_{14}|} + \sqrt{1 + \left(\frac{M^{(\nu)}_{11}}{2|M^{(\nu)}_{14}|}\right)^2}\,, \\ Y &= \frac{m_{\nu_5} - M^{(\nu)}_{22}}{|M^{(\nu)}_{25}|} = -\frac{M^{(\nu)}_{22}}{2|M^{(\nu)}_{25}|} + \sqrt{1 + \left(\frac{M^{(\nu)}_{22}}{2|M^{(\nu)}_{25}|}\right)^2}\,. \end{aligned} \tag{85}$$

Note that always $0 < X \leq 1$ and $0 < Y \leq 1$.

The flavor neutrino fields ν_α are connected to the mass neutrino fields ν_i through the five–dimensional unitary transformation

$$\nu_\alpha = \sum_i (V^\dagger)_{\alpha i}\, \nu_i \tag{86}$$

with $\left(V^\dagger\right)_{\alpha i} = (V)^*_{i\alpha} = V^*_{i\alpha}$, where $V = (V_{i\alpha})$ denotes the lepton 5×5 counterpart of Cabibbo—Kobayashi—Maskawa matrix:

$$V = U^{(\nu)\,\dagger} U^{(e)}\ \ ,\ \ U^{(e)} = \left(U^{(e)}_{\alpha\beta}\right) = \left(\begin{array}{cc} U^{(e)}_{\alpha\beta}\ \ (\alpha, \beta = 1, 2, 3) & 0 \\ 0 & \delta_{\alpha\beta}\ \ (\alpha, \beta = 4, 5) \end{array}\right)\,, \tag{87}$$

where $\left(U^{(e)}_{\alpha\beta}\ \ (\alpha,\, \beta = 1, 2, 3)\right)$ is the charged–lepton diagonalizing unitary matrix given perturbatively in Eq. (4). If there $\alpha^{(e)}/\mu^{(e)}$ (jointly with its numerical coefficients) is neglected, then $U^{(e)} \simeq (\delta_{\alpha\beta})$ and so, we can put in Eq. (86)

$$V_{i\alpha}^{*} = \left(V^{\dagger}\right)_{\alpha i} = \left(U^{(e)\,\dagger}U^{(\nu)}\right)_{\alpha i} \simeq \left(U^{(\nu)}\right)_{\alpha i} = U_{\alpha i}^{(\nu)} \; . \tag{88}$$

In our model, $U_{\alpha i}^{(\nu)}$ are given as in Eq. (84).

6. Neutrino oscillations and their possible damping

Having once found the extended Cabibbo—Kobayashi—Maskawa matrix V, we can calculate the probabilities $P(\nu_\alpha \to \nu_\beta)$ of neutrino oscillations $\nu_\alpha \to \nu_\beta$ (in the vacuum) *i.e.*, the probabilities of (vacuum) oscillations of the flavor neutrino states $|\nu_\alpha\rangle \to |\nu_\beta\rangle$, where $|\nu_\alpha\rangle = \nu_\alpha^{\dagger}|0\rangle$ and

$$|\nu_\alpha\rangle = \sum_i |\nu_i\rangle V_{i\alpha} \tag{89}$$

with $|\nu_i\rangle = \nu_i^{\dagger}|0\rangle$. If allowing that, in general, not all mass neutrino states $|\nu_i\rangle$ are *absolutely* stable, then

$$|\nu_i(t)\rangle = e^{-i(H-i\Gamma)t}|\nu_i\rangle = |\nu_i\rangle e^{-i(E_i - i\gamma_i)t} \; , \tag{90}$$

where $E_i = \sqrt{\vec{p}^{\,2} + m_{\nu_i}^2} \simeq |\vec{p}| + m_{\nu_i}^2/2|\vec{p}|$ and $\gamma_i = (|m_{\nu_i}|/E)\,\gamma_i^{(0)}$ are neutrino energies and decay widths (with $\gamma_i^{(0)}$ and $E \simeq |\vec{p}|$ denoting the neutrino decay widths at rest and neutrino beam energy, respectively). Thus, generally, we obtain for neutrinos (in the vacuum) the following *damped* oscillation formulae:

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) &= |\langle\nu_\beta|e^{-i(H-i\Gamma)t}|\nu_\alpha\rangle|^2 = \sum_{j\,i} V_{j\beta}V_{j\alpha}^{*}V_{i\beta}^{*}V_{i\alpha}e^{i(E_j-E_i)t}e^{-(\gamma_j+\gamma_i)t} \\ &= \delta_{\beta\alpha} + \sum_{j\,i} V_{j\beta}V_{j\alpha}^{*}V_{i\beta}^{*}V_{i\alpha}\left[e^{i(E_j-E_i)t}e^{-(\gamma_j+\gamma_i)t} - 1\right] \; . \end{aligned} \tag{91}$$

They are analogues of the formulae for $K^0 \to \overline{K}^0$ and $\overline{K}^0 \to K^0$ oscillations. Note that Eqs. (91) imply the probability sum rules in the nonunitarity form

$$\sum_\beta P(\nu_\alpha \to \nu_\beta) = \sum_i |V_{i\alpha}|^2 e^{-2\gamma_i t} \; , \tag{92}$$

in spite of the unitarity of V. Of course, the rhs of Eq. (92) is equal to 1, if all (here involved) γ_i are zero. In this case, the damping in Eqs. (91) disappears and they become the *conventional* neutrino oscillation formulae. The same is true for the next Eqs. (93).

If the quartic products in Eqs. (91) are real (as it turns out to be in our case), we can rewrite these equations in the form

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) &= \sum_{j\,i} V_{j\beta} V^*_{j\alpha} V^*_{i\beta} V_{i\alpha} e^{-(\gamma_j+\gamma_i)t} \\ &- \sum_{j>i} V_{j\beta} V^*_{j\alpha} V^*_{i\beta} V_{i\alpha} \sin^2\left(\frac{E_j - E_i}{2}\, t\right) e^{-(\gamma_j+\gamma_i)t} \ , \end{aligned} \tag{93}$$

where the first term is equal to

$$\delta_{\beta\alpha} - \sum_{j\,i} V_{j\beta} V^*_{j\alpha} V^*_{i\beta} V_{i\alpha} \left[1 - e^{-(\gamma_j+\gamma_i)t}\right] \ . \tag{94}$$

Writing $(E_j - E_i)t = \Delta m^2_{j\,i} L/2E$ and $(\gamma_j + \gamma_i)t = (|m_{\nu_j}|\gamma^{(0)}_j + |m_{\nu_i}|\gamma^{(0)}_i)L/E$ with $\Delta m^2_{j\,i} \equiv m^2_{\nu_j} - m^2_{\nu_i}$, $E = |\vec{p}|$ and $L = t$, and then expressing the neutrino masses m_{ν_i} and rest widths $\gamma^{(0)}_i$ in eV, the experimental baseline L in km and the neutrino beam energy in GeV, we can insert

$$\begin{aligned} \frac{E_j - E_i}{2}\, t &\to 1.27 \frac{\Delta m^2_{j\,i} L}{E} \equiv x_j - x_i \ , \\ (\gamma_j + \gamma_i)t &\to 5.07 \frac{(|m_{\nu_j}|\gamma^{(0)}_j + |m_{\nu_i}|\gamma^{(0)}_i)L}{E} \equiv y_j + y_i \end{aligned} \tag{95}$$

in Eq. (91) and (93) (here, $c = 1 = \hbar$)[†].

From Eqs. (93) with $V_{i\alpha} = U^{(\nu)\,*}_{\alpha i}$, we derive in the case of our form (84) of $U^{(\nu)}_{\alpha i}$ the following damped oscillation formulae for active neutrinos ν_e , ν_μ , ν_τ (in the vacuum):

[†] The insertion $L = vt$ with $v = |\vec{p}|/E \simeq c$ $(c = 1)$ is called by Lipkin [10] the "right handwaving" which converts the "gedanken oscillation experiment" *in time* into the real oscillation experiment *in space*. In the first experiment, a flavor neutrino is created by a weak–interaction source (of size $\ll L$) in a momentum eigenstate $|\nu_\alpha, \vec{p}\rangle$ being a superposition of a few energy eigenstates $|\nu_i, E_i\rangle$ (with $E_i = \sqrt{\vec{p}^{\,2} + m^2_{\nu_i}}$) describing mass neutrinos evolving in time. Inversely, in the second experiment, the flavor neutrino is emitted in an energy eigenstate $|\nu_\alpha, E\rangle$ given as a superposition of a few momentum eigenstates $|\nu_i, \vec{p}_i\rangle$ (with $|\vec{p}_i| = \sqrt{E^2 - m^2_{\nu_i}}$) describing mass neutrinos propagating in space (the requirement of coherence within this superposition leads to the condition $|\,|\vec{p}_i| - |\vec{p}_j|\,| \ll 1/\text{source size}$). In the first case $E_i - E_j \simeq \Delta m^2_{ij}/2|\vec{p}|$, while in the second $|\vec{p}_i| - |\vec{p}_j| \simeq \Delta m^2_{ij}/2E$. Here, $E \simeq c|\vec{p}|$ $(c = 1)$. A "wrong handwaving" would be the insertion $L = v_i t_i$ with $v_i = \vec{p}/E_i$.

$$
\begin{aligned}
P(\nu_e \to \nu_\mu) &= 0 = P(\nu_\mu \to \nu_e)\ , \\
P(\nu_e \to \nu_\tau) &= 0 = P(\nu_\tau \to \nu_e)\ , \\
P(\nu_\mu \to \nu_\tau) &= 0 = P(\nu_\tau \to \nu_\mu)\ , \\
P(\nu_e \to \nu_e) &= \left(\frac{e^{-y_1} + X^2 e^{-y_4}}{1+X^2}\right)^2 - \left(\frac{2X}{1+X^2}\right)^2 \sin^2(x_4 - x_1) e^{-(y_4+y_1)}\ , \\
P(\nu_\mu \to \nu_\mu) &= \left(\frac{e^{-y_2} + Y^2 e^{-y_5}}{1+Y^2}\right)^2 - \left(\frac{2Y}{1+Y^2}\right)^2 \sin^2(x_5 - x_2) e^{-(y_5+y_2)}\ , \\
P(\nu_\tau \to \nu_\tau) &= e^{-2y_3}
\end{aligned}
\tag{96}
$$

and those where, beside ν_e , ν_μ , ν_τ, the sterile neutrinos $\nu_s^{(e)}$, $\nu_s^{(\mu)}$ participate explicitly:

$$
\begin{aligned}
P(\nu_e \to \nu_s^{(e)}) &= \left(\frac{X(e^{-y_1} - e^{-y_4})}{1+X^2}\right)^2 + \left(\frac{2X}{1+X^2}\right)^2 \sin^2(x_4 - x_1) e^{-(y_4+y_1)}\ , \\
P(\nu_e \to \nu_s^{(\mu)}) &= 0\ , \\
P(\nu_\mu \to \nu_s^{(e)}) &= 0\ , \\
P(\nu_\mu \to \nu_s^{(\mu)}) &= \left(\frac{Y(e^{-y_2} - e^{-y_5})}{1+Y^2}\right)^2 + \left(\frac{2Y}{1+Y^2}\right)^2 \sin^2(x_5 - x_2) e^{-(y_5+y_2)}\ , \\
P(\nu_\tau \to \nu_s^{(e)}) &= 0\ , \\
P(\nu_\tau \to \nu_s^{(\mu)}) &= 0\ .
\end{aligned}
\tag{97}
$$

The probabilities (96) and (97) satisfy the sum rules (92) which now read :

$$
\begin{aligned}
P(\nu_e \to \nu_e) + P(\nu_e \to \nu_s^{(e)}) &= \frac{e^{-2y_1} + X^2 e^{-2y_4}}{1+X^2}\ , \\
P(\nu_\mu \to \nu_\mu) + P(\nu_\mu \to \nu_s^{(\mu)}) &= \frac{e^{-2y_2} + Y^2 e^{-2y_5}}{1+Y^2}\ .
\end{aligned}
\tag{98}
$$

Note that damping in our neutrino oscillation formulae decreases with growing neutrino energy E, because $y_i = 5.07|m_{\nu_i}|\gamma_i^{(0)} L/E$ decreases. Thus, the larger ν_α–neutrino energy is explored in ν_α–neutrino experiments, the smaller damping influence is exerted on $P(\nu_\alpha \to \nu_\alpha)$, provided not all (involved) γ_i are zero. Of course, the effect of damping, if any, is expected to be very small.

7. A mechanism of negligible damping

Now, we turn to the discussion of a possible mechanism of neutrino instability *i.e.,* instability of mass neutrino states. To this end observe that the neutrino weak current

$$J^{(\nu)\,\mu} = \overline{\nu_{eL}}\gamma^\mu \nu_{eL} + \overline{\nu_{\mu L}}\gamma^\mu \nu_{\mu L} + \overline{\nu_{\tau L}}\gamma^\mu \nu_{\tau L} \;, \tag{99}$$

though it is diagonal in the active neutrinos ν_e , ν_μ , ν_τ, is no longer diagonal in the mass neutrinos ν_1 , ν_2 , ν_3 , ν_4 , ν_5, if the sterile neutrinos $\nu_s^{(e)}$, $\nu_s^{(\mu)}$ really exist. In fact, inserting in Eq. (99) the unitary transformation (86), we obtain generally, beside $\overline{\nu_{iL}}\gamma^\mu \nu_{iL}$, some nondiagonal products $\overline{\nu_{iL}}\gamma^\mu \nu_{jL}$ $(i \neq j)$, since only three of five products $\overline{\nu_{\alpha L}}\gamma^\mu \nu_{\alpha L}$ are originally present in Eq. (99).

For instance, in the case of our form (84) of $U^{(\nu)}_{\alpha i}$, the unitary transformation (86) with $V^*_{i\alpha} = U^{(\nu)}_{\alpha i}$ gives

$$\begin{aligned}
\nu_e &= \frac{1}{\sqrt{1+X^2}}\left(\nu_1 - X\nu_4 e^{i\varphi^{(\nu)}}\right) \;, \\
\nu_\mu &= \frac{1}{\sqrt{1+Y^2}}\left(\nu_2 - Y\nu_5 e^{i\varphi^{(\nu)}}\right) \;, \\
\nu_\tau &= \nu_3 \;, \\
\nu_s^{(e)} &= \frac{1}{\sqrt{1+X^2}}\left(X\nu_1 e^{-i\varphi^{(\nu)}} + \nu_4\right) \;, \\
\nu_s^{(\mu)} &= \frac{1}{\sqrt{1+Y^2}}\left(Y\nu_2 e^{-i\varphi^{(\nu)}} + \nu_5\right) \;.
\end{aligned} \tag{100}$$

Thus, in our case, the neutrino weak current (93) transits into the form

$$\begin{aligned}
J^{(\nu)\,\mu} &= \frac{1}{1+X^2}\left[\overline{\nu_{1L}}\gamma^\mu \nu_{1L} + X^2\overline{\nu_{4L}}\gamma^\mu \nu_{4L} - X\left(\overline{\nu_{1L}}\gamma^\mu \nu_{4L} e^{\varphi^{(\nu)}} + \overline{\nu_{4L}}\gamma^\mu \nu_{1L} e^{-i\varphi^{(\nu)}}\right)\right] \\
&\quad + \overline{\nu_{3L}}\gamma^\mu \nu_{3L} \\
&\quad + \frac{1}{1+Y^2}\left[\overline{\nu_{2L}}\gamma^\mu \nu_{2L} + Y^2\overline{\nu_{5L}}\gamma^\mu \nu_{5L} - Y\left(\overline{\nu_{2L}}\gamma^\mu \nu_{5L} e^{\varphi^{(\nu)}} + \overline{\nu_{5L}}\gamma^\mu \nu_{2L} e^{-i\varphi^{(\nu)}}\right)\right] \;.
\end{aligned} \tag{101}$$

Since in the Standard Model lagrangian this neutrino weak current is coupled to the Z boson [with the coupling constant $-g/(2\cos\theta_W) = -e/(2\sin\theta_W\cos\theta_W)$], some neutrino decays of the type $\nu_i \rightarrow \nu_j\,\nu_k\,\bar{\nu}_l$ with $(i,j) = (1,4)$ or $(4,1)$ and $(2,5)$ or $(5,2)$, and

with similar (k,l), are Z–mediated, so that they can be real processes if only $|m_{\nu_i}| > |m_{\nu_j}| + |m_{\nu_k}| + |m_{\nu_l}|$ (here, $\bar{\nu}_l$ denotes an antiparticle of ν_l).

In the case of our neutrino mass spectrum (83), we get the inequalities $m_{\nu_1} > |m_{\nu_4}|$, $m_{\nu_2} > |m_{\nu_5}|$ and $m_{\nu_2} > m_{\nu_1}$, where in the last relation we make use of $M^{(\nu)}_{22} > M^{(\nu)}_{11}$. Further, $|m_{\nu_5}| > m_{\nu_1}$, $|m_{\nu_5}| > |m_{\nu_4}|$, $m_{\nu_3} > m_{\nu_2}$ and $m_{\nu_3} > |m_{\nu_5}|$, if $Y - X > M^{(\nu)}_{11}/|M^{(\nu)}_{25}|$, $Y > X$, $Y < (M^{(\nu)}_{33} - M^{(\nu)}_{22})/|M^{(\nu)}_{25}|$ and $Y < M^{(\nu)}_{33}/|M^{(\nu)}_{25}|$, respectively. Thus, for $Y - X > M^{(\nu)}_{11}/|M^{(\nu)}_{25}|$ and $Y < (M^{(\nu)}_{33} - M^{(\nu)}_{22})/|M^{(\nu)}_{25}|$ all these inequalities hold. In this case, therefore,

$$|m_{\nu_4}| < m_{\nu_1} < |m_{\nu_5}| < m_{\nu_2} < m_{\nu_3} \;, \tag{102}$$

showing that then $|m_{\nu_4}|$ is the lowest neutrino mass.

We can see that for any virtual decay $\nu_1 \rightarrow \nu_4\, \nu_k\, \bar{\nu}_l$ we get

$$\begin{aligned} m_{\nu_1} - |m_{\nu_4}| - |m_{\nu_k}| - |m_{\nu_l}| &\leq m_{\nu_1} - |m_{\nu_4}| - 2|m_{\nu_4}| = 2M^{(\nu)}_{11} - \sqrt{M^{(\nu)\,2}_{11} + 4|M^{(\nu)}_{14}|^2} \\ &= M^{(\nu)}_{11} - 2|M^{(\nu)}_{14}|X > \text{ or } \leq 0 \,, \end{aligned} \tag{103}$$

depending on $X <$ or $\geq M^{(\nu)}_{11}/2|M^{(\nu)}_{14}|$. This implies that, *a priori*, the decay width of ν_1 neutrino may be $\gamma_1 \neq 0$ or $\gamma_1 = 0$, respectively. Since $|m_{\nu_4}| < m_{\nu_1}$, no virtual decay $\nu_4 \rightarrow \nu_1\, \nu_k\, \bar{\nu}_l$ can be a real process, what leads to $\gamma_4 = 0$ for ν_4 neutrino.

Similarly, for any virtual decay $\nu_2 \rightarrow \nu_5\, \nu_k\, \bar{\nu}_l$, we obtain

$$\begin{aligned} m_{\nu_2} - |m_{\nu_5}| - |m_{\nu_k}| - |m_{\nu_l}| &\leq m_{\nu_2} - |m_{\nu_5}| - 2|m_{\nu_4}| = M^{(\nu)}_{11} + M^{(\nu)}_{22} - \sqrt{M^{(\nu)\,2}_{11} + 4|M^{(\nu)}_{14}|^2} \\ &= M^{(\nu)}_{22} - 2|M^{(\nu)}_{14}|X > 0 \end{aligned} \tag{104}$$

if $X < M^{(\nu)}_{22}/2|M^{(\nu)}_{14}|$, where $M^{(\nu)}_{22} = (4/9)(80/\varepsilon^{(\nu)} - 1)M^{(\nu)}_{11}$ with $\varepsilon^{(\nu)} < 1$ (*cf.* Eq. (1) with $f = \nu$). If true, this gives a nonzero decay width $\gamma_2 \neq 0$ for ν_2 neutrino. On the other hand, for ν_5 neutrino $\gamma_5 = 0$, since $|m_{\nu_5}| < m_{\nu_2}$.

Anticipating that $\gamma_1 = 0$ (or is extremely small) and putting $\gamma_3 = \gamma_4 = \gamma_5 = 0$, we obtain from Eqs. (96) and (97) the following neutrino oscillation formulae (possibly damped if $\gamma_2 \neq 0$):

$$
\begin{aligned}
P(\nu_e \to \nu_e) \; &= \; 1 - \left(\frac{2X}{1+X^2}\right)^2 \sin^2(x_4 - x_1) = 1 - P(\nu_e \to \nu_s^{(e)}) \; , \\
P(\nu_\mu \to \nu_\mu) \; &= \; \left(\frac{e^{-y_2} + Y^2}{1+Y^2}\right)^2 - \left(\frac{2Y}{1+Y^2}\right)^2 \sin^2(x_5 - x_2) = \frac{e^{-2y_2} + Y^2}{1+Y^2} - P(\nu_\mu \to \nu_s^{(\mu)}) \; , \\
P(\nu_\tau \to \nu_\tau) \; &= \; 1 \; .
\end{aligned} \tag{105}
$$

Here,

$$
x_1 - x_4 = 2.53 \frac{|M_{14}^{(\nu)}| \, M_{11}^{(\nu)} L}{E} \quad , \quad x_2 - x_5 = 2.53 \frac{|M_{25}^{(\nu)}| \, M_{22}^{(\nu)} L}{E} \; . \tag{106}
$$

From the neutrino mass spectrum (83) and the definitions (85) of X and Y, we can derive the useful equations expressing $M_{11}^{(\nu)}$ and $|M_{14}^{(\nu)}|$ through X and Δm_{14}^2, as well as $M_{22}^{(\nu)}$ and $|M_{25}^{(\nu)}|$ through Y and Δm_{25}^2:

$$
M_{11}^{(\nu)} = \left(\frac{1 - X^2}{1 + X^2} \Delta m_{14}^2\right)^{1/2} \quad , \quad |M_{14}^{(\nu)}| = \left(\frac{X^2}{1 - X^4} \Delta m_{14}^2\right)^{1/2} \tag{107}
$$

as well as

$$
M_{22}^{(\nu)} = \left(\frac{1 - Y^2}{1 + Y^2} \Delta m_{25}^2\right)^{1/2} \quad , \quad |M_{25}^{(\nu)}| = \left(\frac{Y^2}{1 - Y^4} \Delta m_{25}^2\right)^{1/2} \; . \tag{108}
$$

Further, writing

$$
1 \geq \left(\frac{2X}{1+X^2}\right)^2 \equiv \sin^2 2\theta^{(e)} \quad , \quad 1 \geq \left(\frac{2Y}{1+Y^2}\right)^2 \equiv \sin^2 2\theta^{(\mu)} \; , \tag{109}
$$

we obtain

$$
1 \geq X \equiv \tan \theta^{(e)} \quad , \quad 1 \geq Y \equiv \tan \theta^{(\mu)} \; , \tag{110}
$$

where $0 \leq 2\theta^{(e)} \leq \pi/2$ and $0 \leq 2\theta^{(\mu)} \leq \pi/2$. We can see from Eqs. (108) that for a fixed finite $|M_{25}^{(\nu)}|$ we get $\Delta m_{25}^2 \to 0$ as $Y \to 1$, excluding in this limit the corresponding neutrino oscillations. On the other hand, if we insist in an argument to keep Δm_{25}^2 fixed and nonzero as $Y \to 1$, we formally have $|M_{25}^{(\nu)}| \to \infty$, implying $m_{\nu_2} \to \infty$ and $|m_{\nu_5}| \to \infty$. (In both cases $M_{22}^{(\nu)} \to 0$ as $Y \to 1$.) Analogical conclusions follow from Eqs. (107) for $|M_{14}^{(\nu)}|$ and Δm_{14}^2 (and $M_{11}^{(\nu)}$) when $X \to 1$.

The first Eq. (105) enables us to ascribe the observed deficit of solar ν_e's to $\nu_e \to \nu_s^{(e)}$ oscillations. In fact, we can determine our parameters $M_{11}^{(\nu)}$ and $|M_{14}^{(\nu)}|$ putting

$$\left(\frac{2X}{1+X^2}\right)^2 = \sin^2 2\theta_{\rm solar} \sim 0.75 \;\; , \;\; \Delta m_{14}^2 = \Delta m_{\rm solar}^2 \sim 6.5 \times 10^{-11}\ {\rm eV}^2 \; , \tag{111}$$

if the global vacuum fit to solar data [5] is chosen. Then, due to Eqs. (110) and (107)

$$X = \tan\theta_{\rm solar} \sim 1/\sqrt{3} = 0.577 \;\; , \;\; M_{11}^{(\nu)} \sim 5.70 \times 10^{-6}\ {\rm eV} \;\; , \;\; |M_{14}^{(\nu)}| \sim 4.94 \times 10^{-6}\ {\rm eV}\ . \tag{112}$$

Here, we can see that $M_{11}^{(\nu)}/2|M_{14}^{(\nu)}| = (1-X^2)/2X \sim 1/\sqrt{3} \sim X$. Thus, the condition leading to $\gamma_1 = 0$ is satisfied on the edge [*cf.* Eq. (103)]. At the same time, this shows that the condition $M_{22}^{(\nu)}/2|M_{14}^{(\nu)}| > X$, providing $\gamma_2 \neq 0$ in the second Eq. (105), is fulfilled comfortably [*cf.* Eq. (104)].

Damping in the second Eq. (105) complicates our discussion, though it is natural to expect that this formula allows us to ascribe the observed deficit of atmospheric ν_μ's to $\nu_\mu \to \nu_s^{(\mu)}$ oscillations. In fact, anticipating that damping in this case is tiny [*cf.* Eq. (119)], we may write $\exp(-y_2) \simeq 1 - y_2$ and, therefore,

$$P(\nu_\mu \to \nu_\mu) \simeq 1 - \left(\frac{2Y}{1+Y^2}\right)^2 \sin^2(x_5 - x_2) - y_2 \left(\frac{2Y}{1+Y^2}\right)^2 \left[\frac{1}{2} - \sin^2(x_5 - x_2)\right] \; , \tag{113}$$

where the coefficient at y_2 in the correction $O(y_2)$ is almost compensated to zero. Thus, we can put approximately

$$\left(\frac{2Y}{1+Y^2}\right)^2 \simeq \sin^2 2\theta_{\rm atm} \sim 0.82 \;\text{ to }\; 1 \;\; , \;\; \Delta m_{25}^2 \simeq \Delta m_{\rm atm}^2 \sim (0.5 \;\text{ to }\; 6) \times 10^{-3}\ {\rm eV}^2 \; , \tag{114}$$

where the recent data from Super–Kamiokande atmospheric neutrino experiment [4] is applied. Here, we will put, for instance, $\sin^2 2\theta_{\rm atm} \sim 0.999$ and $\Delta m_{\rm atm}^2 \sim 5 \times 10^{-3}\ {\rm eV}^2$ as in Section 3. Then,

$$Y \simeq \tan\theta_{\rm atm} \sim 0.969 \;\; , \;\; M_{22}^{(\nu)} \sim 0.126 \times 10^{-1}\ {\rm eV} \;\; , \;\; |M_{25}^{(\nu)}| \sim 1.99 \times 10^{-1}{\rm eV} \tag{115}$$

due to Eqs (110) and (108).

Making use of the estimations (112) and (115), we can evaluate $\varepsilon^{(\nu)}$ and $\mu^{(\nu)}$ from Eq. (1) (with $f = \nu$),

$$\begin{aligned} \varepsilon^{(\nu)} &= \frac{80}{1 + 9M_{22}^{(\nu)}/4M_{11}^{(\nu)}} \sim 1.61 \times 10^{-2} \, , \\ \mu^{(\nu)} &= \frac{29M_{11}^{(\nu)}}{\varepsilon^{(\nu)}} \sim 1.03 \times 10^{-2} \text{ eV} \, , \end{aligned} \tag{116}$$

and then, the neutrino masses m_{ν_1}, m_{ν_4}, m_{ν_2}, m_{ν_5} and m_{ν_3} from Eqs. (83),

$$\begin{aligned} m_{\nu_1} &\sim 8.55 \times 10^{-6} \text{ eV} \;\; , \;\; m_{\nu_4} \sim -2.85 \times 10^{-6} \text{ eV} \, , \\ m_{\nu_2} &\sim 2.05 \times 10^{-1} \text{ eV} \;\; , \;\; m_{\nu_5} \sim -1.93 \times 10^{-1} \text{ eV} \end{aligned} \tag{117}$$

and

$$m_{\nu_3} = \frac{\mu^{(\nu)}}{29} \frac{24}{25} \left(624 + \varepsilon^{(\nu)} \right) \sim 2.12 \times 10^{-1} \text{ eV} \, . \tag{118}$$

These masses satisfy consistently the inequalities (102) and reproduce the experimental values (101) and (114): $\Delta m_{14}^2 \sim 6.5 \times 10^{-11}$ eV^2 and $\Delta m_{25}^2 \sim 5 \times 10^{-3}$ eV^2.

Now, we can evaluate the total decay width at rest, $\gamma_i^{(0)}$, for a mass neutrino ν_i decaying through the Z–mediated processes $\nu_i \rightarrow \nu_j \, \nu_k \, \bar{\nu}_l$, where $m_i = E_j + E_k + E_l > m_j + m_k + m_l$ with $m_n = |m_{\nu_n}|$. In the case of m_2, m_5 and $m_2 - m_5$ dominating over m_k and m_l (k, $l = 1, 4$), we obtain the approximate formula

$$\gamma_2^{(0)} = \frac{1}{4} \frac{G_F^2}{192\pi^3} \left(\frac{Y}{1+Y^2} \right)^2 (m_2 - m_5)^4 \, (m_2 + 2m_5) \;\; , \tag{119}$$

where the total decay width $\gamma_2^{(0)}$ is the sum of four partial decay widths for $\nu_2 \rightarrow \nu_5 \, \nu_k \, \bar{\nu}_l$ with $(k \, , \, l) = (1 \, , \, 4) \, , \, (4 \, , \, 1) \, , \, (1 \, , \, 1) \, , \, (4 \, , \, 4)$ which are proportional to

$$\left(\frac{Y}{1+Y^2} \right)^2 \left(\frac{X}{1+X^2} \right)^2 , \; \left(\frac{Y}{1+Y^2} \right)^2 \left(\frac{X}{1+X^2} \right)^2 , \; \left(\frac{Y}{1+Y^2} \right)^2 \left(\frac{1}{1+X^2} \right)^2 , \; \left(\frac{Y}{1+Y^2} \right)^2 \left(\frac{X^2}{1+X^2} \right)^2 ,$$

respectively, the sum of these weights being equal to $Y^2/(1+Y^2)^2$. In this calculation, we used the Standard Model coupling of the neutrino weak current (101) to the Z boson [with

the coupling constant $-g/(2\cos\theta_W)$, where $G_F/\sqrt{2} = g^2/(8M_W) = g^2/(8M_Z\cos\theta_W)]$, and considered the situation when $(p_2 - p_5) \ll M_Z^2$ at the rest frame of decaying ν_2: $p_2 = (m_2\,,\,\vec{0})$. In Eq. (119), the factor 1/4 at the front is a consequence of using the neutral weak current (rather than charged weak current), while $Y^2/(1+Y^2)^2$ stems from mixing of active and sterile neutrinos.

If Y, m_2 and m_5 are estimated as in Eqs. (115) and (117), then the formula (119) gives (with the Fermi constant $G_F = 1.17 \times 10^{-5}$ GeV^{-2}) the extremely small value

$$\gamma_2^{(0)} \sim 10^{-59}\ \mathrm{eV} \tag{120}$$

corresponding to the enormous lifetime $\tau_2 = 1/\gamma_2^{(0)} \sim 10^{43}$ sec (as eV$^{-1} = 6.58\times10^{-16}$ sec). This implies for the Super–Kamiokande atmospheric experiment that $y_2 = 5.07 m_2\gamma_2^{(0)} L/E \sim 10^{-55}$ with $m_2\gamma_2^{(0)} \sim 10^{-60}$, $L \sim 1.3 \times 10^4$ and $E \sim 1$ expressed in eV^2, km and GeV, respectively. Thus, practically, $y_2 = 0$ and so $\exp(-y_2) = 1$. If $m_2 = m_{\nu_2}$ and $m_5 = |m_{\nu_5}|$ grow by one order of magnitude (what is the case when $\sin 2\theta_{\mathrm{atm}}$ rises to 0.9999 and so, Y to 0.990), then $\gamma_2^{(0)}$ becomes not larger than $\sim 10^{-54}$ eV and τ_2 not smaller than $\sim 10^{38}$ sec.

Concluding the last Section, we can say that damping in neutrino oscillation formulae can be completely neglected, unless there are *other* sources of neutrino instability [11], more *effective* than the Z–mediated decays $\nu_i \to \nu_j\,\nu_k\,\bar{\nu}_l$ considered in this paper. The last decays appear in the Standard Model framework if, additionally, there are sterile neutrinos mixing with the active ones and so, breaking the elektroweak symmetry $SU(2) \times U(1)$. Our discussion shows that the neutrino decay widths γ_i are zero for $i = 1, 3, 4, 5$ and are completely negligible for $i = 2$. However, our damped oscillation formulae (93) [and their more specific versions given in Eqs. (96) and (97)] can work for any sort of potential neutrino instability.

Appendix: Majorana sterile neutrinos

The flavor neutrinos, three active ν_e , ν_μ , ν_τ and two sterile $\nu_s^{(e)}$, $\nu_s^{(\mu)}$, considered in Sections 5, 6 and 7, lead to five mass neutrinos ν_1 , ν_2 , ν_3 , ν_4 , ν_5 having pure Dirac masses (also in previous Sections neutrinos had always pure Dirac masses). Now, assume that there are *solely* three active flavor neutrinos, *but* they possess the "Majorana" 2×2 mass matrices

$$\widehat{M}_\alpha^{(\nu)} = \begin{pmatrix} m_\alpha^{(L)} & m_\alpha^{(D)} \\ m_\alpha^{(D)} & m_\alpha^{(R)} \end{pmatrix} \;\; (\alpha = e \,,\, \mu \,,\, \tau) \;\; , \tag{A.1}$$

each consisting of *one* Dirac and *two* Majorana masses, $m_\alpha^{(D)}$ and $m_\alpha^{(L,R)}$, respectively [12]. The mass matrices (A.1) imply the following mass term in the lagrangian:

$$-\mathcal{L}_{\rm mass} = \frac{1}{2} \sum_\alpha \left(\overline{\nu_\alpha^{(a)}} \quad \overline{\nu_\alpha^{(s)}} \right) \, \widehat{M}_\alpha^{(\nu)} \, \begin{pmatrix} \nu_\alpha^{(a)} \\ \nu_\alpha^{(s)} \end{pmatrix} \;\; , \tag{A.2}$$

where

$$\nu_\alpha^{(a)} \equiv \nu_{\alpha\, L} + \left(\nu_{\alpha\, L}\right)^c \, , \, \nu_\alpha^{(s)} \equiv \nu_{\alpha\, R} + \left(\nu_{\alpha\, R}\right)^c \;\; (\alpha = e \,,\, \mu \,,\, \tau) \tag{A.3}$$

are the Majorana flavor neutrinos, three active $\nu_\alpha^{(a)}$ and three sterile $\nu_\alpha^{(s)}$, built up of chiral fields $\nu_{\alpha\, L}$, $\left(\nu_{\alpha\, L}\right)^c = \left(\nu_\alpha\right)_R^c$ and $\nu_{\alpha\, R}$, $\left(\nu_{\alpha\, R}\right)^c = \left(\nu_\alpha\right)_L^c$ involved already in the Dirac flavor neutrinos $\nu_\alpha = \nu_{\alpha\, L} + \nu_{\alpha\, R}$ and antineutrinos $\nu_\alpha^c = (\nu_{\alpha\, L})^c + (\nu_{\alpha\, R})^c$. These *conventional* Majorana sterile neutrinos $\nu_\alpha^{(s)}$ contain, therefore, *no extra* neutrino degrees of freedom, in contrast to our previous Dirac sterile neutrinos $\nu_s^{(e,\mu)} = \nu_{s\, L}^{(e,\mu)} + \nu_{s\, R}^{(e,\mu)}$ involving *extra* chiral fields $\nu_{s\, L}^{(e,\mu)}$ and $\nu_{s\, R}^{(e,\mu)}$. Of course, in contrast to the Dirac, the Majorana neutrinos mix (maximally) the lepton number L.

In the case of "Majorana" mass matrices (A.1), the overall neutrino mass matrix takes the 6×6 form

$$\widehat{M}^{(\nu)} = \left(\delta_{\alpha\beta} \widehat{M}_\alpha^{(\nu)} \right) = \left(\delta_{\alpha\beta} \begin{pmatrix} m_\alpha^{(L)} & m_\alpha^{(D)} \\ m_\alpha^{(D)} & m_\alpha^{(R)} \end{pmatrix} \right) \;\; . \tag{A.4}$$

In this "pure–Majorana" mass matrix there is no mixing between flavor neutrinos from three lepton families $\alpha = e \,,\, \mu \,,\, \tau$.

Diagonalizing the "pure–Majorana" mass matrix (A.4), we obtain the neutrino masses

$$m_\alpha^{I,II} = \frac{m_\alpha^{(L)} + m_\alpha^{(R)}}{2} \mp \sqrt{\left(\frac{m_\alpha^{(L)} + m_\alpha^{(R)}}{2}\right)^2 + m_\alpha^{(D)\,2}} \simeq \frac{m_\alpha^{(L)} + m_\alpha^{(R)}}{2} \mp m_\alpha^{(D)} \tag{A.5}$$

corresponding to six Majorana mass neutrinos

$$\begin{aligned} \nu_\alpha^I &= \cos\theta_\alpha \nu_\alpha^{(a)} - \sin\theta_\alpha \nu_\alpha^{(s)} \ , \\ \nu_\alpha^{II} &= \sin\theta_\alpha \nu_\alpha^{(a)} + \cos\theta_\alpha \nu_\alpha^{(s)} \ , \end{aligned} \tag{A.6}$$

where

$$\begin{aligned} \cos\theta_\alpha &= \frac{m_\alpha^{(D)}}{\sqrt{m_\alpha^{(D)\,2} + \left(m_\alpha^{II} - m_\alpha^{(R)}\right)^2}} \simeq \frac{1}{\sqrt{2}}\left(1 - \frac{m_\alpha^{(L)} - m_\alpha^{(R)}}{4m_\alpha^{(D)}}\right) \simeq \frac{1}{\sqrt{2}} \ , \\ \sin\theta_\alpha &= \frac{m_\alpha^{II} - m_\alpha^{(R)}}{\sqrt{m_\alpha^{(D)\,2} + \left(m_\alpha^{II} - m_\alpha^{(R)}\right)^2}} \simeq \frac{1}{\sqrt{2}}\left(1 + \frac{m_\alpha^{(L)} - m_\alpha^{(R)}}{4m_\alpha^{(D)}}\right) \simeq \frac{1}{\sqrt{2}} \end{aligned} \tag{A.7}$$

with $\theta_\alpha \simeq \pi/4 + \left(m_\alpha^{(L)} - m_\alpha^{(R)}\right)/4m_\alpha^{(D)} \simeq \pi/4$ (m_α^I may be negative). Here, the approximate equalities are valid in the case of $m_\alpha^{(L)} \simeq m_\alpha^{(R)}$. If in addition $m_\alpha^{(L)} \simeq m_\alpha^{(R)} \simeq m_\alpha^{(D)}$, then Eqs. (A.5) give $m_\alpha^I \simeq 0$ and $m_\alpha^{II} \simeq 2m_\alpha^{(D)}$. In contrast, if $m_\alpha^{(L)} \simeq m_\alpha^{(R)} \ll m_\alpha^{(D)}$, they imply $m_\alpha^{I,II} \simeq \mp m_\alpha^{(D)}$ (this case is known as the pseudo–Dirac case). Note that in the case of $m_\alpha^{(L)} \simeq m_\alpha^{(R)}$ the mass neutrinos ν_α^I and ν_α^{II} are in an obvious analogy to the mesons $K_L = pK^0 - q\overline{K^0}$ and $K_S = qK^0 + p\overline{K^0}$, where $q/p \simeq 1 - 2\tilde{\varepsilon} \simeq 1$ is a counterpart of our $\tan\theta_\alpha \simeq 1 - \left(m_\alpha^{(R)} - m_\alpha^{(L)}\right)/2m_\alpha^{(D)} \simeq 1$.

Any model with $m_\alpha^{(L)} \simeq m_\alpha^{(R)}$, leading to the nearly maximal mixing $\nu_\alpha^{I,II} \simeq \left(\nu_\alpha^{(a)} \mp \nu_\alpha^{(s)}\right)/\sqrt{2}$, is orthogonal to the popular see–saw model with $m_\alpha^{(L)} \ll m_\alpha^{(D)} \ll m_\alpha^{(R)}$ which gives $\nu_\alpha^I \simeq \nu_\alpha^{(a)}$ and $\nu_\alpha^{II} \simeq \nu_\alpha^{(s)}$. In fact, in this case we get from Eqs. (A.5) and (A.7)

$$m_\alpha^I \simeq -\frac{m_\alpha^{(D)\,2}}{m_\alpha^{(R)}} \simeq 0 \ , \ m_\alpha^{II} \simeq m_\alpha^{(R)} + \frac{m_\alpha^{(D)\,2}}{m_\alpha^{(R)}} \simeq m_\alpha^{(R)} \tag{A.8}$$

and

$$\cos\theta_\alpha \simeq 1 - \frac{1}{2}\left(\frac{m_\alpha^{(D)}}{m_\alpha^{(R)}}\right)^2 \simeq 1 \ , \ \sin\theta_\alpha \simeq \frac{m_\alpha^{(D)}}{m_\alpha^{(R)}} \simeq 0 \ . \tag{A.9}$$

In both cases, however, we may have very small m^I_α. Notice that the present experimental limit on the (still not observed) neutrinoless double β decay (violating the lepton number L) allows for $m^{(L)}_e$ of the order of 1 eV or smaller in both cases of $m^{(L)}_e \simeq m^{(R)}_e$ and $m^{(L)}_e \ll m^{(R)}_e$.

With the use of the neutrino mass matrix (A.4) we get the "pure–Majorana" oscillation formulae

$$P\left(\nu^{(a)}_\alpha \to \nu^{(s)}_\beta\right) = |\langle\nu^{(s)}_\beta|e^{-iHt}|\nu^{(a)}_\alpha\rangle|^2 = \delta_{\beta\,\alpha} \sin^2 2\theta_\alpha \sin^2\left(x^{II}_\alpha - x^I_\alpha\right) \tag{A.10}$$

and

$$P\left(\nu^{(a)}_\alpha \to \nu^{(a)}_\beta\right) = |\langle\nu^{(a)}_\beta|e^{-iHt}|\nu^{(a)}_\alpha\rangle|^2 = \delta_{\beta\,\alpha} - P\left(\nu^{(a)}_\alpha \to \nu^{(s)}_\beta\right) \quad , \tag{A.11}$$

where $x^{I,\,II}_\alpha = 1.27(m^{I,\,II}_\alpha)^2 L/E$ with $m^{I,\,II}_\alpha$, L and E expressed in eV, km and GeV, respectively. Here, $\sin^2 2\theta_\alpha \simeq 1$ if $m^{(L)}_\alpha \simeq m^{(R)}_\alpha$.

For a form of neutrino mass matrix more general than the "pure–Majorana" form (A.4), more general mass spectrum and mixing appear. A fairly general mixing may be given by the following anticipated formulae for Majorana mass neutrinos:

$$\nu^{I,\,II}_i = \sum_\alpha U^{(\nu)*}_{\alpha\,i}\,\nu^{I,\,II}_\alpha = \sum_\alpha U^{(\nu)*}_{\alpha\,i} \left\{ \begin{array}{l} \cos\theta_\alpha\,\nu^{(a)}_\alpha - \sin\theta_\alpha\,\nu^{(s)}_\alpha \\ \sin\theta_\alpha\,\nu^{(a)}_\alpha + \cos\theta_\alpha\,\nu^{(s)}_\alpha \end{array}\right. \tag{A.12}$$

(with $i = 1, 2, 3$ and $\alpha = e\,,\,\mu\,,\,\tau$). Here $U^{(\nu)} = \left(U^{(\nu)}_{\alpha\,i}\right)$ is a 3×3 family unitary matrix diagonalizing a 3×3 neutrino family mass matrix $M^{(\nu)} = \left(M^{(\nu)}_{\alpha\beta}\right)$ through the relation $\left(U^{(\nu)\,\dagger}M^{(\nu)}U^{(\nu)}\right)_{ij} = \delta_{ij}m_i$.

If the Majorana mixing angle θ_α is taken as a universal θ (what certainly would be the case for $\theta_\alpha = 45^\circ$ corresponding to $m^{(L)}_\alpha = m^{(R)}_\alpha$), then the mixing (A.12) follows from the 6×6 neutrino mass matrix

$$\widehat{M}^{(\nu)} = \left(\widehat{M}^{(\nu)}_{\alpha\beta}\right) \;\;\text{with}\;\; \widehat{M}^{(\nu)}_{\alpha\beta} = M^{(\nu)}_{\alpha\beta} \left(\begin{array}{cc} \lambda^{(L)} & \lambda^{(D)} \\ \lambda^{(D)} & \lambda^{(R)} \end{array}\right) \quad , \tag{A.13}$$

all entries $\lambda^{(L)}$, $\lambda^{(R)}$ and $\lambda^{(D)}$ being dimensionless. In fact, such a form leads to the 6×6 unitary matrix

$$\widehat{U}^{(\nu)} = \left(\widehat{U}^{(\nu)}_{\alpha\,i}\right) \;\;\text{with}\;\; \widehat{U}^{(\nu)}_{\alpha\,i} = U^{(\nu)}_{\alpha\,i} \left(\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array}\right) \tag{A.14}$$

which diagonalizes $\widehat{M}^{(\nu)}$ according to the relation

$$\left(\widehat{U}^{(\nu)\,\dagger}\widehat{M}^{(\nu)}\widehat{U}^{(\nu)}\right)_{ij} = \delta_{ij}\left(\begin{array}{cc} m_i^I & 0 \\ 0 & m_i^{II}\end{array}\right) \quad , \tag{A.15}$$

where

$$m_i^{I,\,II} = m_i\lambda^{I,\,II} \;\;\text{with}\;\; \lambda^{I,\,II} = \frac{\lambda^{(L)}+\lambda^{(R)}}{2} \mp \sqrt{\left(\frac{\lambda^{(L)}-\lambda^{(R)}}{2}\right)^2 + \lambda^{(D)\,2}} \simeq \frac{\lambda^{(L)}+\lambda^{(R)}}{2} \mp \lambda^{(D)} \tag{A.16}$$

($i = 1,2,3$) are neutrino masses. The approximate equality in Eq. (A.16) is valid for $\lambda^{(L)} \simeq \lambda^{(R)}$. Note that the mass matrix (A.13) is the direct product of two matrices (3×3 and 2×2) containing separately the family and "Majorana" degrees of freedom. Thus, also the spectrum (A.16) is multiplicative.

In the case of neutrino mass matrix (A.13), the "pure–Majorana" oscillation formulae (A.11) are extended to the form

$$\begin{aligned} P\left(\nu_\alpha^{(a)} \rightarrow \nu_\beta^{(a)}\right) &= |\langle\nu_\beta^{(a)}|e^{-iHt}|\nu_\alpha^{(a)}\rangle|^2 = \delta_{\beta\,\alpha} - \sin^2 2\theta \sum_i |U^{(\nu)}_{\beta\,i}|^2 |U^{(\nu)}_{\alpha\,i}|^2 \sin^2\left(x_i^{II} - x_i^I\right) \\ & -4\sum_{j>i} U^{(\nu)\,*}_{\beta\,j} U^{(\nu)}_{\alpha\,j} U^{(\nu)}_{\beta\,i} U^{(\nu)\,*}_{\alpha\,i} \left\{\cos^4\theta \sin^2\left(x_i^I - x_j^I\right) + \sin^4\theta\sin^2\left(x_j^{II} - x_i^{II}\right)\right. \\ & \left. + \cos^2\!\theta\sin^2\!\theta\left[\sin^2\left(x_j^{II} - x_i^I\right) + \sin^2\left(x_j^I - x_i^{II}\right)\right]\right\} \end{aligned} \tag{A.17}$$

which holds when the quartic products of matrix elements $U^{(\nu)}_{\alpha\,i}$ are real. In Eqs. (A.17), $x^{I,\,II} = 1.27(m_i^{I,\,II})^2 L/E$. Here, $\sin^2 2\theta \simeq 1$ and $\cos^2\theta \simeq 1/2 \simeq \sin^2\theta$ if $\lambda^{(L)} \simeq \lambda^{(R)}$.

The neutrino family mass matrix $M^{(\nu)} = \left(M^{(\nu)}_{\alpha\beta}\right)$ may be assumed in the form (1) (with $f = \nu$). Then, in the case of small $\xi = M^{(\nu)}_{33}/|M^{(\nu)}_{12}|$ and $\chi = M^{(\nu)}_{22}/|M^{(\nu)}_{12}|$, the family unitary matrix $U^{(\nu)} = \left(U^{(\nu)}_{\alpha i}\right)$ is given in Eqs. (9). In order to derive from the neutrino oscillation formulae (A.17) explicit results, we put $\lambda^{(L)} = \lambda^{(R)}$ $\left(\equiv \lambda^{(M)}\right)$. In this case, the neutrino mass matrix (A.13) has the form

$$\widehat{M}^{(\nu)} = \left(\widehat{M}^{(\nu)}_{\alpha\beta}\right) \;\;\text{with}\;\; \widehat{M}^{(\nu)}_{\alpha\beta} = M^{(\nu)}_{\alpha\beta}\left(\begin{array}{cc} \lambda^{(M)} & \lambda^{(D)} \\ \lambda^{(D)} & \lambda^{(M)}\end{array}\right) \quad , \tag{A.18}$$

and the neutrino mass spectrum gives $m_i^{I,\,II} = m_i\left(\lambda^{(M)} \mp \lambda^{(D)}\right)$, where $m_i \equiv m_{\nu_i}$ are determined as in Eqs. (5) implying $m_3 \gtrsim |m_2| \gg m_1$ ($m_2 = -|m_2|$).

With this mass spectrum, the further discussion depends on the ratio of $\lambda^{(M)}$ and $\lambda^{(D)}$. We will consider two cases: (i) $\lambda^{(M)} = \lambda^{(D)}$ or (ii) $\lambda^{(M)} \ll \lambda^{(D)}$ (the pseudo–Dirac case). We derive from Eqs. (A.17) and (9) the following neutrino oscillation formulae: in the case (i)

$$\begin{aligned}
P\left(\nu_e^{(a)} \to \nu_e^{(a)}\right) &= 1 - \frac{48}{49}\sin^2\left(1.27\frac{4m_1^2\lambda^{(D)\,2}L}{E}\right) - \frac{97}{2\cdot 49^2}\ , \\
P\left(\nu_\mu^{(a)} \to \nu_\mu^{(a)}\right) &= 1 - \sin^2\left(1.27\frac{4m_2^2\lambda^{(D)\,2}L}{E}\right)\ , \\
P\left(\nu_\mu^{(a)} \to \nu_e^{(a)}\right) &= \frac{1}{4\cdot 49}\sin^2\left(1.27\frac{4(m_3^2-m_2^2)\lambda^{(D)\,2}L}{E}\right)
\end{aligned} \tag{A.19}$$

or, in the case (ii)

$$\begin{aligned}
P\left(\nu_e^{(a)} \to \nu_e^{(a)}\right) &= 1 - \left(\frac{48}{49}\right)^2\sin^2\left(1.27\frac{4m_1^2\lambda^{(M)}\lambda^{(D)}L}{E}\right) - \frac{387}{4\cdot 49^2}\ , \\
P\left(\nu_\mu^{(a)} \to \nu_\mu^{(a)}\right) &= 1 - \frac{1}{2}\sin^2\left(1.27\frac{4m_2^2\lambda^{(M)}\lambda^{(D)}L}{E}\right) - \sin^2\left(1.27\frac{4(m_3^2-m_2^2)\lambda^{(D)\,2}L}{E}\right)\ , \\
P\left(\nu_\mu^{(a)} \to \nu_e^{(a)}\right) &= \frac{1}{49}\sin^2\left(1.27\frac{4(m_3^2-m_2^2)\lambda^{(D)\,2}L}{E}\right) - \frac{1}{2\cdot 49}\sin^2\left(1.27\frac{4m_2^2\lambda^{(M)}\lambda^{(D)}L}{E}\right)\ ,
\end{aligned} \tag{A.20}$$

where the L's are three different experimental baselines. In these equations, the negligible constant terms come out from terms containing $\sin^2$ of large phases averaged over many oscillation lengths determined by the leading terms with $\sin^2$ of small phases. The phases in Eqs. (A.19) and (A.20) were calculated in both cases from the relations

$$\begin{aligned}
(m_j^{I,II})^2 - (m_i^{I,II})^2 &= m_j^2\left(\lambda^{(M)} \mp \lambda^{(D)}\right)^2 - m_i^2\left(\lambda^{(M)} \mp \lambda^{(D)}\right)^2\ , \\
(m_j^{II,I})^2 - (m_i^{I,II})^2 &= m_j^2\left(\lambda^{(M)} \pm \lambda^{(D)}\right)^2 - m_i^2\left(\lambda^{(M)} \mp \lambda^{(D)}\right)^2\ ,
\end{aligned} \tag{A.21}$$

working for $\lambda^{(L)} = \lambda^{(R)}\ \left(\equiv \lambda^{(M)}\right)$. Note that the second and third Eq. (A.20) are not of the two–flavor form, in contrast to the second and third Eq. (A.19).

Comparing two first oscillation formulae (A.19) with the results of solar and atmospheric neutrino experiments [*cf.* Eqs (111) and (114)], respectively, we get

$$\frac{48}{49} \leftrightarrow \sin^2 2\theta_{\rm sol} \sim 0.75 \;\;,\;\; 4m_1^2 \lambda^{(D)\,2} \leftrightarrow \Delta m^2_{\rm sol} \sim 6.5 \times 10^{-11}\;{\rm eV}^2 \tag{A.22}$$

and

$$1 \leftrightarrow \sin^2 2\theta_{\rm atm} \sim 0.82 \;\;{\rm to}\;\; 1 \;\;,\;\; 4m_2^2 \lambda^{(D)\,2} \leftrightarrow \Delta m^2_{\rm atm} \sim (0.5\;\;{\rm to}\;\;6) \times 10^{-3}\;{\rm eV}^2\;. \tag{A.23}$$

Hence, we obtain

$$\frac{m_1}{|m_2|} \sim (3.61\;\;{\rm to}\;\;1.04) \times 10^{-4} \tag{A.24}$$

and, due to Eqs. (5),

$$\xi = (49)^{3/2} \frac{m_1}{|m_2|} \sim (12.4\;\;{\rm to}\;\;3.57) \times 10^{-2}\;, \tag{A.25}$$

while $m_3^2 - m_2^2 = 14[(48/49)\xi + \chi]|M^{(\nu)}_{12}|^2 \sim (1.80\;\;{\rm to}\;\;0.52)|M^{(\nu)}_{12}|^2$ with $\chi = \xi/16.848$. This estimation confirms that $\xi \equiv M^{(\nu)}_{33}/|M^{(\nu)}_{12}|$ and $\chi \equiv M^{(\nu)}_{22}/|M^{(\nu)}_{12}|$ are small.

In contrast to solar and atmospheric results, the LSND result (*cf.* Ref. [6]), say, $\sin^2 2\theta_{\rm LSND} \sim 0.02$ and $\Delta m^2_{\rm LSND} \sim 0.05$ eV^2 cannot be explained in the case (i), since in the third Eq. (A.19)

$$4(m_3^2 - m_2^2)\lambda^{(D)\,2} \ll 4m_2^2 \lambda^{(D)\,2} \sim (0.5\;\;{\rm to}\;\;6) \times 10^{-3}\;{\rm eV}^2 < \Delta m^2_{\rm LSND} \tag{A.26}$$

for the estimation (A.25) ($m_3^2 \gtrsim m_2^2 \simeq 49|M^{(\nu)}_{12}|$).

In the case (ii), however, one may try to compare the third Eq. (A.20) with the LSND result getting, say,

$$\frac{1}{49} \leftrightarrow \sin^2 2\theta_{\rm LSND} \sim 0.02 \;\;,\;\; (m_3^2 - m_2^2)\lambda^{(D)\,2} \leftrightarrow \Delta m^2_{\rm LSND} \sim 0.05\;{\rm eV}^2\;. \tag{A.27}$$

If in the case (ii) the relation $4m_2^2 \lambda^{(M)} \lambda^{(D)} \leftrightarrow \Delta m^2_{\rm atm}$ analogical to (A.23) holds approximately [*cf.* the second Eq. (A.20)], the comparison with (A.27) gives

$$\frac{4m_2^2 \lambda^{(M)}}{(m_3^2 - m_2^2)\lambda^{(D)}} = \frac{\Delta m^2_{\rm atm}}{\Delta m^2_{\rm LSND}} \sim (0.1\;\;{\rm to}\;\;1.2) \times 10^{-1} \tag{A.28}$$

and

$$\frac{\lambda^{(M)}}{\lambda^{(D)}} = \frac{1}{14}\left(\frac{48}{49}\xi + \chi\right)\frac{\Delta m^2_{\rm atm}}{\Delta m^2_{\rm LSND}} \sim (9.2 \;\; \text{to} \;\; 2.6) \times 10^{-3} \frac{\Delta m^2_{\rm atm}}{\Delta m^2_{\rm LSND}} \sim (0.92 \;\; \text{to} \;\; 3.2) \times 10^{-4}\,, \tag{A.29}$$

since

$$\frac{m_3^2 - m_2^2}{m_2^2} = \frac{2}{7}\left(\frac{48}{49}\xi + \chi\right) \tag{A.30}$$

through Eqs. (5) (in making the estimation (A.29) the value (A.25) was used, which holds also in the case (*ii*) if $4m_1^2\lambda^{(M)}\lambda^{(D)} \leftrightarrow \Delta m^2_{\rm sol}$). Thus, for the value, (A.29) of $\lambda^{(M)}/\lambda^{(D)}$ the third Eq. (A.20) might be consistent with the LSND result.

In conclusion of this Appendix, we can say that a simple neutrino mass matrix (A.13), operating with three neutrinos $\nu_e\,,\,\nu_\mu\,,\,\nu_\tau$ only and being multiplicative in "Majorana" and family degrees of freedom, is consistent in a natural way with solar and atmospheric neutrino experiments, but not with the LSND result (that still requires confirmation). Such a consistency of "Majorana" option does not differ much from that based on the neutrino mass matrix (82) including two Dirac sterile neutrinos $\nu_s^{(e)}$ and $\nu_s^{(\mu)}$. These conclusions were drawn with the use of our family mass matrix (1) (with $f = \nu$), where the dominance of its off–diagonal elements was conjectured. The opposite conjecture of dominance of its diagonal elements does not change our conclusions essentially. The nearly bimaximal mixing that appears in the $\nu_e^{(a)} \rightarrow \nu_e^{(a)}$ and $\nu_\mu^{(a)} \rightarrow \nu_\mu^{(a)}$ oscillation formulae (A.19) is a consequence of maximal mixings of $\nu_e^{(a)}$ with $\nu_e^{(s)}$ and $\nu_\mu^{(a)}$ with $\nu_\mu^{(s)}$, reflecting the equality $\lambda^{(L)} = \lambda^{(R)}$ and so, not holding in the see–saw model corresponding to $\lambda^{(L)} \ll \lambda^{(D)} \ll \lambda^{(R)}$.

When discussing the Majorana flavor neutrinos $\nu_\alpha^{(a)}$ and $\nu_\alpha^{(s)}$ ($\alpha = e,\, \mu,\, \tau$), one presumes that the superpositions (A.3) defining formally these objects are really coherent in processes of electroweak interactions which operate on lefthanded chiral fields $\nu_{\alpha L} = \nu_{\alpha L}^{(a)}$, ignoring their righthanded counterparts $\nu_{\alpha R} = \nu_{\alpha R}^{(s)}$.

The Dirac part of mass term (A.2) and the kinetic term $\sum_\alpha \overline{\nu_\alpha} i\gamma \cdot \partial \nu_\alpha$ can be expressed by ν_α as well as $\nu_\alpha^{(a)}$ and $\nu_\alpha^{(s)}$, *viz.*

$$-\mathcal{L}_{\rm mass}^{(D)} = \sum_{\alpha} m_{\alpha}^{(D)} \overline{\nu_{\alpha}} \nu_{\alpha} = \sum_{\alpha} m_{\alpha}^{(D)} \left(\overline{\nu_{\alpha}^{(s)}} \nu_{\alpha}^{(a)} + \overline{\nu_{\alpha}^{(a)}} \nu_{\alpha}^{(s)} \right) \tag{A.31}$$

and, up to the full divergence $i\partial \cdot \sum_{\alpha} \overline{\nu_{\alpha}} \gamma \nu_{\alpha}$,

$$\mathcal{L}_{\rm kin} = \sum_{\alpha} \overline{\nu_{\alpha}} i\gamma \cdot \partial \nu_{\alpha} = \frac{1}{2} \sum_{\alpha} \left(\overline{\nu_{\alpha}^{(a)}} i\gamma \cdot \partial \nu_{\alpha}^{(a)} + \overline{\nu_{\alpha}^{(s)}} i\gamma \cdot \partial \nu_{\alpha}^{(s)} \right) \ . \tag{A.32}$$

Thus, the deciding role in the coherence question is played by the Majorana part of the mass term (A.2),

$$\begin{aligned} -\mathcal{L}_{\rm mass}^{(D)} &= \frac{1}{2} \sum_{\alpha} \left(m_{\alpha}^{(L)} \overline{\nu_{\alpha}^{(a)}} \nu_{\alpha}^{(a)} + m_{\alpha}^{(R)} \overline{\nu_{\alpha}^{(s)}} \nu_{\alpha}^{(s)} \right) \\ &= \frac{1}{2} \sum_{\alpha} \left\{ m_{\alpha}^{(L)} \left[\overline{(\nu_{\alpha L})^c} \nu_{\alpha L} + \overline{\nu_{\alpha L}} (\nu_{\alpha L})^c \right] + m_{\alpha}^{(R)} \left[\overline{(\nu_{\alpha R})^c} \nu_{\alpha R} + \overline{\nu_{\alpha R}} (\nu_{\alpha R})^c \right] \right\} \ , \end{aligned} \tag{A.33}$$

which can be presented also in terms of Dirac superpositions $\nu_{\alpha} = \nu_{\alpha L} + \nu_{\alpha R}$ and $\nu_{\alpha}^c = (\nu_{\alpha L})^c + (\nu_{\alpha R})^c$, but *only* if $m_{\alpha}^{(L)} = m_{\alpha}^{(R)}$. Hence, if $m_{\alpha}^{(L)} \neq m_{\alpha}^{(R)}$ (or even if $m_{\alpha}^{(L)} \simeq m_{\alpha}^{(R)}$ only approximately), the coherence of Majorana superpositions $\nu_{\alpha}^{(a)}$ and $\nu_{\alpha}^{(s)}$ seems to be physically preferred over the coherence of Dirac superposition ν_{α}.

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